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Development of an elastic-plastic finite element program with the initial strain approach

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DEVELOPMENT OF AN ELASTIC-PLASTIC

FINITE ELEMENT PROGRAM

WITH THE INITIAL STRAIN APPROACH

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ABSTRACT

A finite element method is presented for solution of plane stress and axisymmetric elastic-plastic problems employing the method of successive elastic solutions with the initial strain approach. The problem of nonconvergence of the initial strain approach for perfectly plastic problems was eliminated by expressing the Prandtl-Ruess equations entirely in terms of strain. Two examples are presented to demonstrate the validity of the method.

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NOMENCLATURE AND LIST OF SYMBOLS

The following symbols are used in this presentation. A tilde ($\tilde{\ }$) indicates the quantity is a submatrix in a partitioned matrix,

*,**	- footnote symbols,
$[\ ]$	- matrix of dimensions r x c,
$\{\ \}$	- column matrix (vector) of dimensions r x 1,
$[I]$	- identity matrix,
$[\ ]^T$	- transpose of a matrix,
$[\ ]^{-1}$	- inverse of a square matrix,
$[\bar{\ }]$	- overall matrix formed by combining element matrices,
$\{\bar{\ }\}$	- overall column matrix formed by combining element column matrices,
$\left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$	- a partitioned matrix,
$\left\{\begin{array}{c} \\ \hline \\ \end{array}\right\}$	- a partitioned column matrix,
r,θ,z	- coordinates,
i	- subscript or superscript denoting element number,
M	- total number of elements,
m	- superscript denoting one of four triangles of quadrilateral elements,
N	- maximum number of load increments,
n	- load increment,
ℓ	- subscript points on stress strain curve,
k	- subscript used to indicate iteration number within an increment,

x	- load step size expressed as a multiple of first yield load,
E	- modulus of elasticity,
ν	- Poisson's ratio,
λ_i	- element convergence parameter,
γ	- convergence parameter,
nplas	- convergence index,
P.E.	- potential energy,
U^i	- strain energy of element i,
W^i	- element displacement,
F^i	- element body forces,
P^i	- element surface tractions,
dV_i	- differential of element volume,
dA_i	- differential of element surface area,
$\varepsilon_{ri},\varepsilon_{\theta i},\varepsilon_{zi},\varepsilon_{rzi}$	- engineering strains,
$\sigma_{ri},\sigma_{\theta i},\sigma_{zi},\sigma_{rzi}$	- engineering stresses,
$\varepsilon'_{ri},\varepsilon'_{\theta i},\varepsilon'_{zi},\varepsilon'_{rzi}$	- modified engineering strains,
$\varepsilon''_{ri},\varepsilon''_{\theta i},\varepsilon''_{zi},\varepsilon''_{rzi}$	- elastic portion of engineering strains,
ε_{eti}	- total equivalent strain,
ε'_{eti}	- modified total strain,
ε_{eei}	- equivalent elastic strain of a uniaxial test,
$\varepsilon_{xi},\sigma_{xi}$	- engineering strain and engineering stress from a uniaxial test(in the direction of the load),
$\varepsilon_{\ell},\sigma_{\ell}$	- coordinates of a point on the material stress strain curve which are input to the computer as data,
σ_y	- material yield stress,

σ^{e}_{max} - maximum effective element stress,

σ_{ei} - element effective stress,

$\sigma_{ei,k}$ - equivalent stress for element i for iteration k

$\Delta\varepsilon_{pi,k}$ - estimate of plastic strain for element i for iteration k,

$\left(\frac{d\sigma_e}{d\varepsilon_{pi}}\right)_k$ - slope of the equivalent stress, plastic strain curve at $\sigma_{ei,k}$,

$\Delta\varepsilon_{pi}$ - element increment of plastic strain for the last load increment,

$\Delta\sigma_{ei}$ - increment of element effective stress,

$\{U\}_i$ - element nodal point displacements,

$[d]_i$ - transforms element displacements to nodal point displacements,

$\{\varepsilon\}_i$ - element total strains vector,

$[a]_i$ - strain displacement transformation matrix,

$\{\sigma\}_i$ - element stress vector,

$[c]_i$ - element elasticity matrix,

$\{\varepsilon_o\}_i$ - element plastic and thermal strain combined,

$\{\varepsilon^{e}\}_i$ - element elastic strain component,

$\{\varepsilon^{pt}\}_i$ - element total plastic strain component,

$\{\varepsilon^{t}\}_i$ - element thermal strain component,

$\{\varepsilon^{p}\}_i$ - element plastic strain component for plastic strain that occurred prior to the last increment of load,

$\{\Delta\varepsilon^{p}\}_i$ - plastic strain due to the last increment of load,

$[k]_i$ - element stiffness matrix,

$[\bar{K}]$ - overall stiffness matrix,

$\{q\}_i$ - element combined load vector,

$\{\bar{Q}\}$ - overall combined load vector,

$\{L\}_i$ - element load vector due to body forces, initial strains and thermal strains,

$\{R\}_i$ - element load vector due to element surface tractions,

$\{F\}_i$ - element body force vector,

$\{P\}_i$ - element surface traction vector,

$\{CP\}_i$ - element concentrated nodal point loads,

$\{B\}_i$ - combined element load vector for body forces thermal strains and plastic strain prior to the last increment of load,

$\{b\}_i$ - element load vector due to plastic strain of the last load increment,

$\{\overline{C_{load}}\}$ - vector of loads due to body forces, surface tractions, concentrated nodal point loads and plastic strain for the load prior to the last increment of load,

$\{\overline{D_{elb}}\}$ - vector of loads due to plastic strain resulting from the last increment of load,

$\{\overline{Incrmt}\}$ - increment of total load,

$\{\overline{Q_{yield}}\}$ - combined load vector which produces yield in only the highest stressed element,

$\{\bar{Q}\}_n$ - combined load vector for nth increment of load,

$\{\overline{C_{load}}\}_n$ - increment of loads due to body forces, surface tractions, concentrated nodal point loads and plastic strain for the load prior to the last increment of load,

$\{\overline{D_{elb}}\}_n$ - increment of loads due to plastic strain resulting from the last increment of load,

$\{W\}_i$ - element displacements,

$\{\overline{U}\}$ - vector of nodal point displacements for the overall structure,

$(\varepsilon_{yield})_i$ - element yield strain,

$\sigma_{o,i}$ - element yield stress,

$\{\varepsilon'\}_i$ - element modified total strain,

$\varepsilon'_{et,k}$ - modified equivalent strain for k iteration

ε_{pi} - sum of increments of plastic strain prior to the last load increment,

$\Delta\varepsilon^p_{ri},\Delta\varepsilon^p_{\theta i},\Delta\varepsilon^p_{zi},\Delta\varepsilon^p_{rzi}$ - increment of plastic strain,

E_ℓ - slope of uniaxial stress strain curve at ℓ,

j - index of nodal points,

x - load step size as a multiple of first yield load,

$[\overline{K'}]$ - reduced total stiffness matrix,

$\{\overline{Q'}\}$ - reduced total load vector,

$[H]_i$ - portion of strain displacement transformation matrix that is independent of location,

$[H^m]_i$ - $[H]_i$ for one of the four triangles that form a quadrilateral,

$[g]_i$ - portion of strain displacement transformation matrix that is dependent on location,

$[g^m]_i$ - $[g]_i$ for one of the four triangles that form a quadrilateral,

$\{\alpha\}_i$ - intermediate storage vector used to store the product $[H]_i\{U\}_i$,

$\{\alpha 1\}_i$ - first four rows of $\{\alpha\}_i$,

$\{\alpha 2\}_i$ - last two rows of $\{\alpha\}_i$,

$\{U1\}_i$ - first eight rows of element displacement vector,

$\{U2\}_i$ - last two rows of element displacement vector,

$[H1]_i$ - (4 x 8) upper left of $[H]_i$,

$[H2]_i$ - (4 x 2) upper right of $[H]_i$,

$[H3]_i$ - (2 x 8) lower left of $[H]_i$,

$[H4]_i$ - (2 x 2) lower right of $[H]_i$,

$[HH]_i$ - transformation matrix which converts nodal point displacements to element strains,

$\{q1\}_i$ - first eight rows of element load vector,

$\{q2\}_i$ - last two rows of element load vectors,

$[k1]$ - (8 x 8) upper left of element stiffness matrix,

$[k2]$ - (8 x 2) upper right of element stiffness matrix,

$[k3]$ - (2 x 8) lower left of element stiffness matrix,

$[k4]$ - (2 x 2) lower right of element stiffness matrix,

$[AH]_i$ - storage matrix for $[k4]^{-1}$,

$[S]_i$ - storage matrix for $[k3]$,

$[F_a]_i$ - storage matrix which relates the last two nodal loads to $\{\alpha\}_i$,

$[HI]_i$ - storage matrix which relates the first eight nodal displacements to $\{\alpha\}_i$,

$\{AI\}_i$	- storage vector for the product $[q]_i[F_a]_i\{q_2\}_i$,
$[FH]_i$	- transformation matrix which converts element plastic strains to plastic loads,
$[FH1]_i$	- first eight rows of $[FH]_i$,
$[FH2]_i$	- last two rows of $[FH]_i$,
$\{b1\}_i$	- first eight rows of element plastic load vector,
$[EH]_i$	- last two rows of $[FH]_i$,
$[A1P]_i$	- storage matrix for product $[q]_i[F_a]_i[FH2]_i$,
$[D^m]_i$	- storage matrix for $\int_{V_m} [g^m]_i^T \, dV_m [C]_i$
$\{q''\}_i$	- (8 x 1) vector of element loads,
$\{b''\}_i$	- (8 x 1) vector of element plastic loads,
$[A]$	- storage matrix for $[k2][k4]^{-1}[k3]$.

Note: When terms of a partitioned matrix are written outside of an enclosing matrix symbol the tilde (~) will be replaced by the appropriate matrix symbol.

I. INTRODUCTION

Nonlinear elasticity and plasticity have long been of interest to mathematicians and engineers. The development of computers and numerical techniques have provided a means of solution for problems that previously could not be solved. The finite element method is one of the numerical techniques which has evolved for solution of elasticity as well as other engineering problems.

The direct stiffness method of Turner et al. (12)* and the formulation of the initial strain approach by Mendelson and Manson (10) were combined by Gallagher et al. (4) for stress analysis of heated structures. This combination is the finite element method with the initial strain approach, which extended the use of the finite element method to elastic-plastic problems. The finite element method is now generally accepted and will only be outlined in Chapter II. Detailed information on the finite element method is in a text by Zienkiewicz (16). An example of a readily available finite element program is the Sass II computer program. Full development of the Sass II finite element program can be found in the report by Jones et al. (3).

The initial strain approach can be subdivided into the constant stress approach and the constant strain approach:

a) The constant stress approach assumes the stress calculations to be correct and determines plastic strain directly from the stress strain curve.

b) The constant strain approach considers the total strains to be

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*Numbers underlined and in parentheses refer to listings in the bibliography.

correct and separates the total strains into elastic and plastic strains.

Two major defects have been associated with the initial strain approach - nonconvergence for large load increments and nonconvergence for zero strain hardening (5). Marcal (6) proved the initial strain approach would theoretically not converge for a perfectly plastic material.

Because of the early difficulties with the initial strain approach, the tangent modulus or partial stiffness method came into use. The partial stiffness method depends on changing the material properties based on element stresses. This method will solve perfectly plastic problems, but is very dependent on load increment size.

The latest method to appear in the literature is the initial stress method. The initial stress method adjusts element stresses to obtain its solution. Zienkiewicz et al. (15) showed the initial stress method to be independent of load step size and was able to solve perfectly plastic problems.

All the methods (initial strain, partial stiffness, and initial stress) depend on the method of successive approximations. The difference in the three iterative techniques depends on which variable is adjusted to incorporate new approximations into the iteration cycle. The success of the initial stress method indicated that an initial strain approach could be found which would work equally well. Zienkiewicz states, "...even in ideal plasticity increments of strain prescribe uniquely the stress system (while the reverse is not true for ideal plasticity)" (15).

The initial strain approach was chosen for this study because it had all the benefits of an initial process and seemed a more natural approach than that of the initial stress method. The difficulty of nonconvergence for a perfectly plastic material appeared to be resolvable.

II. FINITE ELEMENT EQUATIONS

The complete development of the finite element equations of this chapter can be found in the report by Jones et al. (3). An outline of the procedure will be presented here for reference.

An elastic body is divided into M elements, the potential energy of the body is then expressed as the sum of the potential energy of the elements.

$$P.E. = \sum_{i=1}^{M} \left(U^i - \int_{Vol\ i} W^i F^i dV_i - \int_{Area\ i} W^i P^i dA_i \right) \qquad (2.1)^*$$

where

U^i - element strain energy

F^i - element body forces

P^i - element surface tractions

W^i - element displacement

A displacement field is assumed for the elements and the element displacements are written in terms of nodal point displacements.

$$\{W\}_i = [d]_i \{u\}_i \qquad (2.2)$$

$\{W\}_i$ - element displacement

$\{u\}_i$ - element nodal point displacements

$[d]_i$ - necessary matrix to relate element displacements to nodal point displacement

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*Numbers in parentheses refer to equations

**Matrix notations will be employed in this thesis with symbolic representation defined on page vii.

An expression for element strain can be obtained directly from the strain displacement equations and written in matrix form as,

$$\{\varepsilon\}_i = [a]_i\{u\}_i \tag{2.3}$$

The stresses are then related to the element strains as,

$$\{\sigma\}_i = [c]_i\{\varepsilon\}_i - [c]_i\{\varepsilon_o\}_i \tag{2.4}$$

where

$[c]_i$ - element elasticity matrix

$\{\varepsilon_o\}_i$ - combined plastic and thermal strain vector

The total potential energy can be written in terms of nodal point displacements.

$$P.E. = \frac{1}{2}\{\overline{U}\}^T[\overline{K}]\{\overline{U}\} - \{\overline{U}\}^T\{\overline{Q}\} \tag{2.5}$$

where

$[\overline{K}]$ - stiffness matrix for the overall structure

$\{\overline{U}\}$ - vector of the nodal point displacements for the overall structure

$\{\overline{Q}\}$ - vector of loads for the entire structure

The overall stiffness matrix $[\overline{K}]$ can be found from

$$\overline{K} = \sum_{i=1}^{M}[k]_i \tag{2.6}$$

where

$$[k]_i = \int_{\text{Vol } i} [a]_i^T [c]_i [a]_i \, dV_i \tag{2.7}$$

The load vector $\{\overline{Q}\}$ can be found from

$$\{\overline{Q}\} = \sum_{i=1}^{M} \{L\}_i + \sum_{i=1}^{M} \{R\}_i + \sum_{i=1}^{M} \{CP\}_i \tag{2.8}$$

or

$$\{\overline{Q}\} = \sum_{i=1}^{M} \{q\}_i$$

where

$\{L\}_i$ - element load vector due to body forces, initial strains and thermal strains

$\{R\}_i$ - element load vector due to element surface tractions

$\{CP\}_i$ - concentrated nodal point forces

The element load vectors $\{L\}_i$ and $\{R\}_i$ are defined as

$$\{L\}_i = \int_{\text{Vol } i} \left([d]_i^T \{F\}_i + [a]_i^T [c]_i \{\varepsilon_o\}_i \right) dV \tag{2.9}$$

and

$$\{R\}_i = \int_{\text{Area } i} [d]_i^T \{P\}_i \, dA \tag{2.10}$$

where

$\{F\}_i$ - element body force vector

$\{P\}_i$ - element surface traction vector

The potential energy is minimized with respect to nodal point displacements in equation (2.5) to obtain

$$\{\bar{Q}\} = [\bar{K}]\{\bar{U}\} \tag{2.11}$$

This equation can be solved for displacements and then the element stress and strain can be calculated.

III. METHOD OF SUCCESSIVE ELASTIC SOLUTIONS WITH THE INITIAL STRAIN APPROACH

The initial strain approach separates the total strain into elastic and plastic components. If the plastic strains were known it would be an easy task to solve for the elastic strains; but the plastic strains are not known at the start of a problem. The method of successive elastic solutions is an iterative procedure used to obtain the plastic strain by combining the initial strain idea with a method of improving estimates of plastic strain and iterating for a solution.

A. The Initial Strain Approach (Constant Strain)

One technique used to solve elastic plastic problems is to consider an elastic deformation followed by a perfectly plastic deformation. This is essentially the initial strain approach except the plastic strain is known or estimated prior to the calculation of the elastic strain. The line ABC of figure 1 shows the development of strain in the initial strain procedure for a given stress condition. The total effective strain, plastic strain, elastic strain and effective stress can be obtained whether the stress strain curve ADC is followed or the path ABC is traversed. For structures which do not have much change in loading path with the development of plastic strains there is little difference between following path ABC or ADC. Small load increments are used to insure an accurate solution when the loading path changes with development of plastic strains.

The first step of the initial strain approach is to separate the strain components into elastic, plastic and thermal strain components

$$\{\varepsilon\}_i = \{\varepsilon^e\}_i + \{\varepsilon^{pt}\}_i + \{\varepsilon^t\}_i \tag{3.1}$$

where

$\{\varepsilon\}_i$ - element total strain vector

$\{\varepsilon^e\}_i$ - element elastic strain component vector

$\{\varepsilon^{pt}\}_i$ - element plastic strain component vector

$\{\varepsilon^t\}_i$ - element thermal strain component vector

Further separating the plastic strain into known components and unknown components gives

$$\{\varepsilon\}_i = \{\varepsilon^e\}_i + \{\varepsilon^p\}_i + \{\Delta\varepsilon^p\}_i + \{\varepsilon^t\}_i \tag{3.2}$$

where

$\{\varepsilon^p\}_i$ - known plastic strain vector, that is, plastic strain that has occurred prior to the application of the last increment of load.

$\{\Delta\varepsilon^p\}_i$ - unknown plastic strain vector, that is, plastic strain due only to the last increment of load.

Equation (3.2) can be written in expanded form for an axisymmetric body as

$$\begin{aligned}
\varepsilon_{ri} &= \frac{\sigma_{ri}}{E} - \frac{\nu\sigma_{\theta i}}{E} - \frac{\nu\sigma_{zi}}{E} + \varepsilon^p_{ri} + \Delta\varepsilon^p_{ri} + \varepsilon^t_{ri} \\
\varepsilon_{\theta i} &= -\frac{\nu\sigma_{ri}}{E} + \frac{\sigma_{\theta i}}{E} - \frac{\nu\sigma_{zi}}{E} + \varepsilon^p_{\theta i} + \Delta\varepsilon^p_{\theta i} + \varepsilon^t_{\theta i} \\
\varepsilon_{zi} &= -\frac{\nu\sigma_{ri}}{E} - \frac{\nu\sigma_{\theta i}}{E} + \frac{\sigma_{zi}}{E} + \varepsilon^p_{zi} + \Delta\varepsilon^p_{zi} + \varepsilon^t_{zi} \\
\varepsilon_{rzi} &= \frac{2(1+\nu)}{E}\sigma_{rzi} + \varepsilon^p_{rzi} + \Delta\varepsilon^p_{rzi}
\end{aligned} \tag{3.2a}$$

B. Yield Criteria

Von Mises yield criteria expressed as strain is used to determine initial and subsequent yield. Yield occurs when

$$\varepsilon_{eti} \geq (\varepsilon_{yield})_i \tag{3.3}$$

where

$$(\varepsilon_{yield})_i = \frac{\sigma_{o,i}}{E} \tag{3.3a}$$

$(\varepsilon_{yield})_i$ - element yield strain

$\sigma_{o,i}$ - element uniaxial yield stress

and

$$\varepsilon_{eti} = \frac{\sqrt{2}}{3}\left[(\varepsilon_{ri}-\varepsilon_{\theta i})^2+(\varepsilon_{\theta i}-\varepsilon_{zi})^2+(\varepsilon_{zi}-\varepsilon_{ri})^2+1.5\varepsilon_{rzi}{}^2\right]^{\frac{1}{2}} \tag{3.3b}$$

C. Modified Total Strains and Method of Estimating Plastic Strain

Mendelson (9) introduces the concept of modified total strain which allows the plastic strain for each load increment to be calculated without calculating the plastic strain for the previous load increments. The modified total strain is defined by

$$\{\varepsilon'\}_i = \{\varepsilon^e\}_i + \{\Delta\varepsilon^p\}_i \tag{3.4}$$

Equation (3.4) can be written in expanded form for an axisymmetric body as

$$\varepsilon'_{ri} = \frac{\sigma_{ri}}{E} - \frac{\nu\sigma_{\theta i}}{E} - \frac{\nu\sigma_{zi}}{E} + \Delta\varepsilon^p_{ri}$$

$$\varepsilon'_{\theta i} = -\frac{\nu\sigma_{ri}}{E} + \frac{\sigma_{\theta i}}{E} - \frac{\nu\sigma_{zi}}{E} + \Delta\varepsilon^p_{\theta i} \tag{3.4a}$$

$$\epsilon'_{zi} = -\frac{\nu\sigma_{ri}}{E} - \frac{\nu\sigma_{\theta i}}{E} + \frac{\sigma_{zi}}{E} + \Delta\epsilon^p_{zi}$$

$$\epsilon'_{rzi} = \frac{2(1+\nu)}{E}\sigma_{rzi} + \Delta\epsilon^p_{rzi}$$

The modified equivalent strain then becomes

$$\epsilon'_{eti} = \frac{\sqrt{2}}{3}\left[(\epsilon'_{ri}-\epsilon'_{\theta i})^2+(\epsilon'_{\theta i}-\epsilon'_{zi})^2+(\epsilon'_{zi}-\epsilon'_{ri})^2+1.5\epsilon'^2_{rzi}\right]^{\frac{1}{2}} \tag{3.5}$$

The equivalent plastic strain increment is developed by Mendelson (9) using a uniaxial stress strain curve. He obtains

$$\Delta\epsilon_{pi,k}{}^{*} = \frac{\epsilon'_{eti,k} - \frac{2}{3}[(1+\nu)/E]\sigma_{ei,k-1}}{1 + \frac{2}{3}[(1+\nu)/E](d\sigma_{ei}/d\epsilon_{pi})_{k-1}} \tag{3.6}$$

where

$\sigma_{ei,k-1}$ - effective stress from previous iteration for $\Delta\epsilon_{pi}$

$\left(\frac{d\sigma_{ei}}{d\epsilon_{pi}}\right)_{k-1}$ - slope of the effective stress, plastic strain curve at $\sigma_{ei,k-1}$

$\Delta\epsilon_{pi,k}$ - estimate of increment of plastic strain for iteration k

Figure 2 shows the relationship between the quantities of equation (3.6) when final values have been obtained.

From figure 2 an equation can be written for the equivalent strain as

$$\epsilon_{eti} = \Delta\epsilon_{pi} + \frac{2}{3}(1+\nu)\frac{\sigma_{ei}}{E} + \epsilon_{pi} \tag{3.7}$$

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*The k subscript indicates intermediate values obtained during iteration.

or the modified equivalent strain as

$$\varepsilon'_{eti} = \Delta\varepsilon_{pi} + \frac{2}{3}(1+\nu)\frac{\sigma_{ei}}{E} \tag{3.8}$$

where

$$\sigma_{ei} = \frac{\sqrt{2}}{2}\left[(\sigma_{ri}-\sigma_{\theta i})^2+(\sigma_{\theta i}-\sigma_{zi})^2+(\sigma_{zi}-\sigma_{ri})^2+6\sigma_{rzi}{}^2\right]^{\frac{1}{2}} \tag{3.9}$$

For uniaxial tension in the z-direction

$$\sigma_{ei} = \sigma_{zi} \tag{3.9a}$$

Once the equivalent plastic strain increment $\Delta\varepsilon_{pi}$ is determined from equation (3.6), the plastic strain $\{\Delta\varepsilon_p\}_i$ can be found from the Prandtl-Reuss equations (9).

$$\begin{aligned}
\Delta\varepsilon^p_{ri} &= \frac{\Delta\varepsilon_{pi}}{3\varepsilon'_{eti}}(2\varepsilon'_{ri} - \varepsilon'_{\theta i} - \varepsilon'_{zi}) \\
\Delta\varepsilon^p_{\theta i} &= \frac{\Delta\varepsilon_{pi}}{3\varepsilon'_{eti}}(2\varepsilon'_{\theta i} - \varepsilon'_{zi} - \varepsilon'_{ri}) \\
\Delta\varepsilon^p_{zi} &= \frac{\Delta\varepsilon_{pi}}{3\varepsilon'_{eti}}(2\varepsilon'_{zi} - \varepsilon'_{ri} - \varepsilon'_{\theta i}) \\
\Delta\varepsilon^p_{rzi} &= \frac{\Delta\varepsilon_{pi}}{\varepsilon'_{eti}}\varepsilon'_{rzi}
\end{aligned} \tag{3.10}$$

D. The Method of Successive Elastic Solutions with the Initial Strain Approach

The solution of an elastic problem consists of solving equation (2.11), equation (2.3) and equation (2.4) for displacements, strains, and stresses, respectively. This is also the first step of the solution of an elastic-plastic problem. The total strains would then be separated into elastic and plastic components by equation (3.6) and equation (3.10). If the total strains were correct then the elastic and plastic components of strain would also be correct. The total strains from an elastic solution could not be correct because the displacements and strains were assumed to be totally elastic when in fact they were partly plastic. The plastic strain obtained from equation (3.10) is then used as an estimate of plastic strain and the original problem becomes an elastic problem with an initial plastic strain. The original problem is solved again using the estimate of plastic strain and a new estimate for the plastic strain is obtained.

The finite element equations are rewritten to reflect the estimate of plastic strain. Equation (2.9) for nodal point loads due to body forces, thermal strains and initial plastic strains can be rewritten as

$$\{L\}_i = \int_{\mathrm{Vol}\ i} \left([d]_i^T \{F\}_i + [a]_i^T [c]_i (\{\varepsilon^t\}_i + \{\varepsilon^p\}_i + \{\Delta\varepsilon^p\}_i) \right) dV_i \qquad (3.11)$$

or as

$$\{L\}_i = \{B\}_i + \{b\}_i \qquad (3.12)$$

where

$$\{B\}_i = \int_{\mathrm{Vol}\ i} \left([d]_i^T \{F\}_i + [a]_i^T [c]_i (\{\varepsilon^t\}_i + \{\varepsilon^p\}_i) \right) dV_i \qquad (3.13)$$

and

$$\{b\}_i = \int_{Vol\ i} [a]_i^T [c]_i \{\Delta\varepsilon^p\}_i dV \tag{3.14}$$

For any particular loading $\{\varepsilon^p\}_i$, $\{\varepsilon^t\}_i$, $\{F\}_i$ are all known and $\{b\}_i$ is the only unknown factor in equation (3.12). For the first increment of load $\{\varepsilon^p\}_i$ is zero. The total load equation (2.8) becomes

$$\{\bar{Q}\} = \sum_{i=1}^{m} \{R\}_i + \sum_{i=1}^{m} \{B\}_i + \sum_{i=1}^{m} \{b\}_i + \sum_{i=1}^{m} \{CP\}_i \tag{3.15}$$

or

$$\{\bar{Q}\} = \{\overline{C_{load}}\} + \{\overline{D_{elb}}\} \tag{3.16}$$

where

$$\{\overline{C_{load}}\} = \sum_{i=1}^{m} \{R\}_i + \sum_{i=1}^{m} \{B\}_i + \sum_{i=1}^{m} \{CP\}_i \tag{3.17}$$

or

$\{\overline{C_{load}}\}$ - nodal point loads due to body forces, surface tractions, concentrated external loads, thermal loads, and initial plastic strain that has occurred prior to the last load increment.

and

$$\{\overline{D_{elb}}\} = \sum_{i=1}^{m} \{b\}_i \tag{3.18}$$

or

$\{\overline{D_{elb}}\}$ - nodal point loads due to an estimate of plastic strain for the last increment of load.

and

$\{CP\}_i$ - element concentrated external loads

Equation (2.11) for nodal point displacements becomes

$$\{\overline{D_{elb}}\} + \{\overline{C_{load}}\} = [\bar{K}]\{\bar{U}\} \tag{3.19}$$

After the elastic solution and plastic strain estimates are obtained plastic loads are calculated and equation (3.19) is solved for displacements. This allows another estimate of plastic strain to be made which continues the iteration cycle. When the values of total equivalent strain, ε_{eti} converge, the iteration is ended.

An outline of this procedure and a flow chart is given below:

1. Set $\{\varepsilon^p\}_i$, $\{\Delta\varepsilon^p\}_i$ and ε_{eti} equal to zero for all elements.
2. Solve equation (2.11) for nodal point displacement and compute element strains and stresses with equation (2.3) and equation (2.4).
3. Equation (3.3b)is solved for the equivalent strain and used with the stress strain curve to determine if an element has yielded.
4. Compare the values of $\varepsilon_{eti,k}$ obtained in step 3 with the values of $\varepsilon_{eti,k-1}$

   $$\text{percent difference} = \frac{\varepsilon_{eti,k} - \varepsilon_{eti,k-1}}{\varepsilon_{eti,k}} \times 100$$

   This comparison is made for each element. The k subscript denotes the iteration number.

   If the percent difference is less than a specified quantity then the iteration is ended for this load increment by going to step 8.

5. For elements that have yielded, equation (3.10) is used to estimate the plastic strain $\{\Delta\varepsilon^p\}_i$.
6. The element plastic load vector $\{b\}_i$ is calculated for elements that yield from equation (3.14) and is combined with equation (3.18) to form the overall plastic load vector.
7. Solve equation (3.19) for nodal point displacements and compute element strains and stresses with equation (2.3) and equation (2.4). Return to step 3.
8. Set $\{\varepsilon^p\}_i$ equal to the sum of the value of $\{\varepsilon^p\}_i$ plus $\{\Delta\varepsilon^p\}_i$. Set $\{\Delta\varepsilon^p\}_i$ equal to zero and add a new load increment to $\{\overline{Q}\}$. Return to step 7 unless the solution for the total load has been obtained. The method of incrementing the load is discussed in Chapter IV.

Figure 3 is a flow chart of the above procedure.

IV. PROGRAM DEVELOPMENT

The Wilson (14) finite element program was modified to solve elastic plastic stress problems by successive elastic solutions with the initial strain approach. A uniform plane strip loaded in uniaxial tension, using the grid shown in Figure 4, was chosen as a test problem. Figure 5 shows a plot of iterations verses total equivalent strain for the test problem. The test problem was fully plastic, i.e., all elements were plastic. Figure 5 demonstrates the program's ability to converge to the exact value of element strain. The strain approaches the exact strain from below which is a necessary factor for convergence. The large number of iterations results from small strain hardening and the requirement for no more than 5×10^{-5} percent change in strain between iterations for convergence.

The second test problem was a perforated tension strip as shown in Figure 6. The grid used in solving this problem is shown in Figure 7. A load equivalent to two times that required for first yield was applied to the strip. The program obtained a solution for this problem, but the stresses and strains of some elements were not on the stress strain curve. Figure 8 illustrates how the stress and strain varied during the iteration to the final solution for four elements. The points crossed the stress strain curve and never returned to the curve but found a new equilibrium position. When a plastic element nearly satisfies the stress strain curve, relief by its own plastic strain plus a change in loading path due to other elements being plastic can cause the element to be over strained. An iterative scheme was built into the computer program to insure no element was over strained. Test problem two was solved again and results indicated that the method worked properly. Test problem two was then solved again for a perfectly plastic material and again reasonable results were obtained.

An incremental loading procedure was incorporated into the program which allows the total load to be applied to the structure in steps.

A. Limiting Plastic Strain

The constant strain approach assumes the calculated total strains for an increment are correct. The elastic and plastic components are then separated as shown in equation (3.1). If too large a component of plastic strain is obtained from equation (3.10) the stress calculated with equation (2.4) will be too low. Two factors can cause the estimate of the unknown plastic strain to be too large.

1) The unloading of an element due to element interaction is not accounted for in the plastic strain estimation.
2) The change in slope of the stress strain curve is neglected when estimating the plastic strain increment.

To insure that an estimate of plastic strain never becomes too large, the uniaxial stress strain curve is used as an upper bound on the strain. This is done by finding an equivalent uniaxial strain for the element and using this strain to compute an equivalent stress from the material stress strain curve. This equivalent stress is then the lower bound of the element stress. A plastic strain estimate is allowed to lower the element stress to a point where it is greater than or equal to the uniaxial equivalent stress.

This procedure depends on finding the uniaxial stress from the equivalent uniaxial strain. To change a uniaxial stress strain curve to an equivalent stress strain curve requires only the modification of the elastic strain component by the factor $(1+\nu)2/3$. Writing the uniaxial strain in terms of effective stress and strain of a uniaxial test gives,

$$\varepsilon_{xi} = \frac{\sigma_{ei}}{E} + \varepsilon_{eti} - \varepsilon_{eei} \tag{4.1}$$

where

ε_{xi} - uniaxial strain of a uniaxial test in the direction of load

ε_{eti} - total equivalent strain of a uniaxial test

ε_{eei} - equivalent elastic strain of a uniaxial test

The substitution for σ_{ei} from

$$\varepsilon_{eei} = \frac{2}{3}\frac{\sigma_{ei}}{E}(1+\nu) \tag{4.2}$$

into Equation (4.1) gives

$$\varepsilon_{xi} = \varepsilon_{eti} - \varepsilon_{eei} \frac{(\nu-.5)}{(\nu+1)} \tag{4.3}$$

where

$$\varepsilon_{eei} = \frac{\sqrt{2}}{3}[(\varepsilon^e_{ri}-\varepsilon^e_{\theta i})^2+(\varepsilon^e_{\theta i}-\varepsilon^e_{zi})^2+(\varepsilon^e_{zi}-\varepsilon^e_{ri})^2+1.5\varepsilon^e_{rzi}{}^2]^{\frac{1}{2}} \tag{4.4}$$

and

$$\{\varepsilon^e\}_i = \{\varepsilon\}_i - \{\varepsilon^p\}_i - \{\Delta\varepsilon^p\}_i \tag{4.5}$$

The strain ε_{xi} is then used to locate a point on the uniaxial stress strain curve. The stress strain curve is made up of five straight line sections as shown in Figure 9. The values of stress and strain are known for five discrete points and the curve is formed by straight lines between these points. Once ε_{xi} is computed the uniaxial stress is found from the equation

$$\sigma_{xi} = \sigma_{\ell} + E_{\ell}\ (\varepsilon_{xi} - \varepsilon_{\ell}) \tag{4.6}$$

where

E_{ℓ} - slope of uniaxial stress strain curve at ℓ

ε_{ℓ} - the largest discrete strain that is less than ε_{xi}

σ_{ℓ} - the discrete stress corresponding to ε_{ℓ}

σ_{xi} - uniaxial stress corresponding to the theoretical uniaxial strain of equation (4.1) and the material stress strain curve

After the uniaxial stress σ_{xi} has been determined the effective stress is computed from

$$\{\sigma\}_i = [c]_i(\{\varepsilon\}_i - \{\varepsilon_o\}_i) \tag{4.7}$$

where

$\{\varepsilon_o\}$ - combined plastic and thermal strain vector

and

$$\sigma_{ei} = \frac{\sqrt{2}}{2}\,[(\sigma_{ri}-\sigma_{\theta i})^2+(\sigma_{\theta i}-\sigma_{zi})^2+(\sigma_{zi}-\sigma_{ri})^2+6\sigma_{rzi}{}^2]^{\frac{1}{2}} \tag{4.8}$$

If $\sigma_{xi} \leq \sigma_{ei}$ then the plastic strain estimate is not so great as to strain the element past the stress strain curve. If $\sigma_{xi} > \sigma_{ei}$ then the element has been over relieved and the estimate of plastic strain is too large. An iterative procedure was incorporated to reduce the plastic strain until $\sigma_{xi} \leq \sigma_{ei}$.

B. Incrementation

The total load is applied to the structure and an elastic solution for stresses obtained. A ratio of the element stress and material yield stress is calculated for the element with the highest effective stress. The total load vector $\{\bar{Q}\}$ is then reduced by this ratio to obtain a load vector which will cause yielding in one element. This load is then used to start the incrementation. The yield load is

$$\{\overline{Q_{yield}}\} = \frac{\sigma_y}{\sigma^e_{max}} \{\bar{Q}\} \tag{4.9}$$

where

$\{\overline{Q_{yield}}\}$ - a constant load which will cause the highest stressed element to yield

σ_y - material yield stress

σ^e_{max} - effective stress of highest stressed element

The total load minus the yield load is divided into multiples of the yield load. The loading for an increment of load then becomes,

$$\{\bar{Q}\}_n = (1+nx)\{\overline{Q_{yield}}\} \tag{4.10}$$

where

$\{\bar{Q}\}_n$ - load vector for nth increment of load

n - increment number

x - load step size as a multiple of first yield load

During an increment while iterating for the plastic loads the total load $\{\bar{Q}\}$ would be,

$$\{\bar{Q}\}_n = \{\overline{C_{load}}\}_n + \{\overline{D_{elb}}\}_n \tag{4.11}$$

C. Additional Test Problems

To insure that the program functioned properly for thermal loading or body force loading, a series of small test problems were solved. In each case the solution obtained was compared to an analytical solution or a solution obtained from another finite element program. In all cases the results obtained from the program of this study were accurate.

V. RESULTS AND CONCLUSIONS

A. Discussion of Results

After the test problems were solved satisfactorily, two example problems were chosen to demonstrate the computer program's capability.

1. Example 1. Example 1 is the perforated tension strip shown in Figure 6 which was experimentally studied by Theocaris and Marketos (13) using birefringent coatings. Marcal and King solved the same problem using a finite element program with the partial stiffness method (7). The grid of Marcal and King was not given in their paper. Zienkiewicz, Valliappant, and King also analyzed this problem using the initial stress method with their finite element program (15). Zienkiewicz used a grid with 97 nodal points and 107 elements similar to the grid of Figure 10, which is the grid used for the present study.

Figure 11 shows the plastic enclaves obtained experimentally by Theocaris and Marketos for the tension strip of Figure 6. Figures 12, 13, and 14 show the plastic enclaves of the initial stress, partial stiffness, and initial strain methods, respectively. Examination of the plastic enclaves shows the method of initial strains approximates the experimental plastic enclaves as well as either the initial stress or partial stiffness methods.

2. Example 2. Example 2 was an axisymmetric cylinder with a 90° v-notch. The material was the same as in example 1 except the material was ideally plastic. Figure 15 shows the notched cylinder and loading for this example. The grid is shown in Figure 16. A small number of

elements was chosen to conserve computer time. Additional elements would be required near the tip of the notch for more accurate results. The plastic enclaves for this example are shown in Figure 17. Marcal and King (7) solved a similar problem but neither the grid nor mechanical properties were given in their paper. The number of elements and nodal points were listed as 250, 150, respectively. The plastic enclaves of the initial strain method have the same shape as those of Marcal and King but occur at different values of loading. The differences could be accounted for by the differences in element grid and material properties. The grid of Figure 16 was too coarse for very accurate results but did give adequate proof of the program's ability to solve perfectly plastic problems.

B. Conclusions

The method of initial strains can be used in conjunction with the finite element method to solve elastic-plastic problems. Large load increments can be taken and convergence obtained. Perfectly plastic material assumptions can be handled as well as actual strain hardening materials.

Although examples with large load increment size were not included, enough test problems were analyzed to show that a relatively large step size could be taken and convergence obtained. The validity of a solution obtained for a large step size would depend on whether small deformation theory was applicable to the problem.

Further work should be done to show the dependence or independence of step size on obtaining a final solution. Also a large and varied number of elastic-plastic problems should be solved to obtain a measure of the limitations of this program.

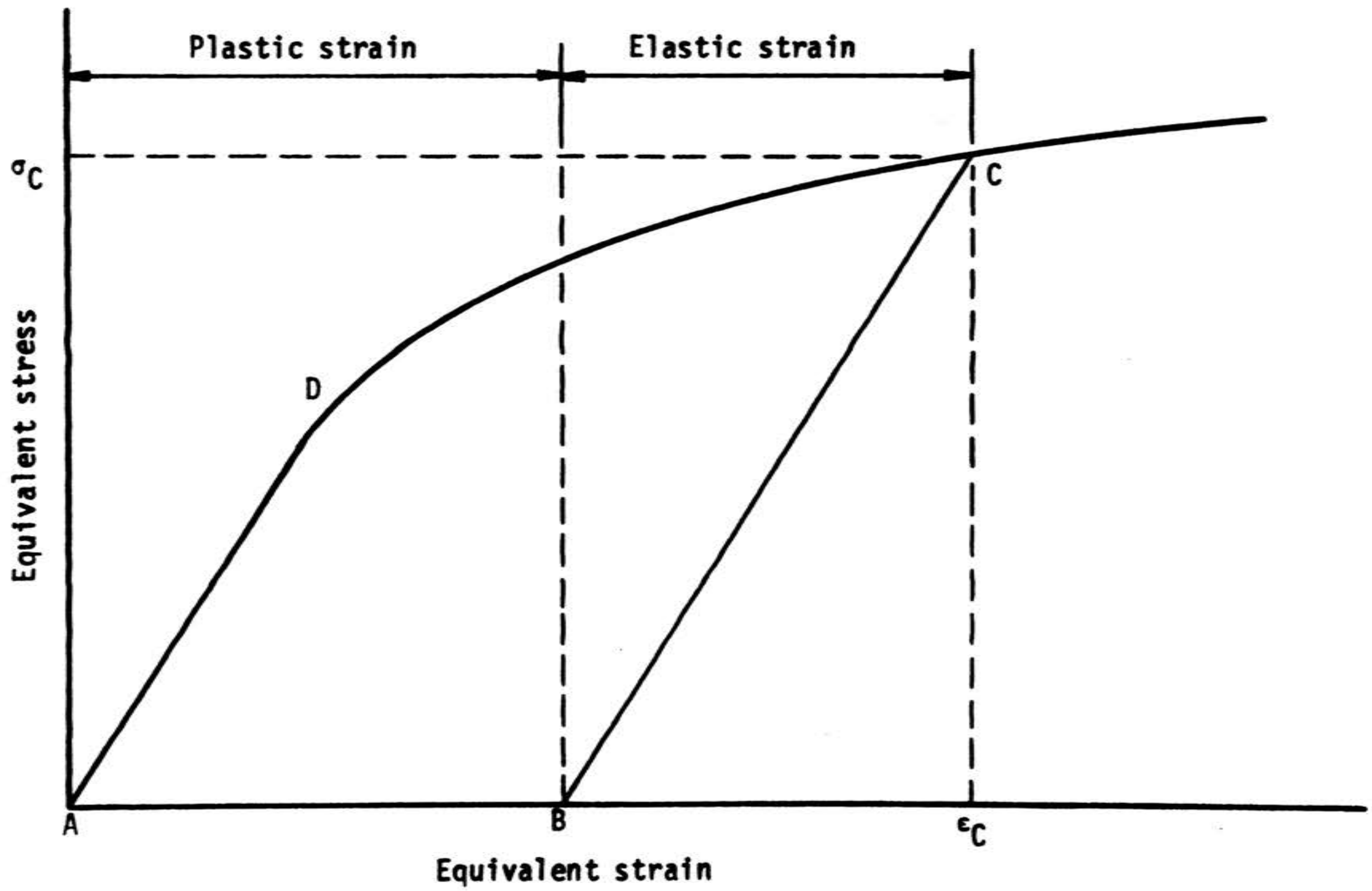

Figure 1. Initial Strain, Separation of Elastic and Plastic Strain

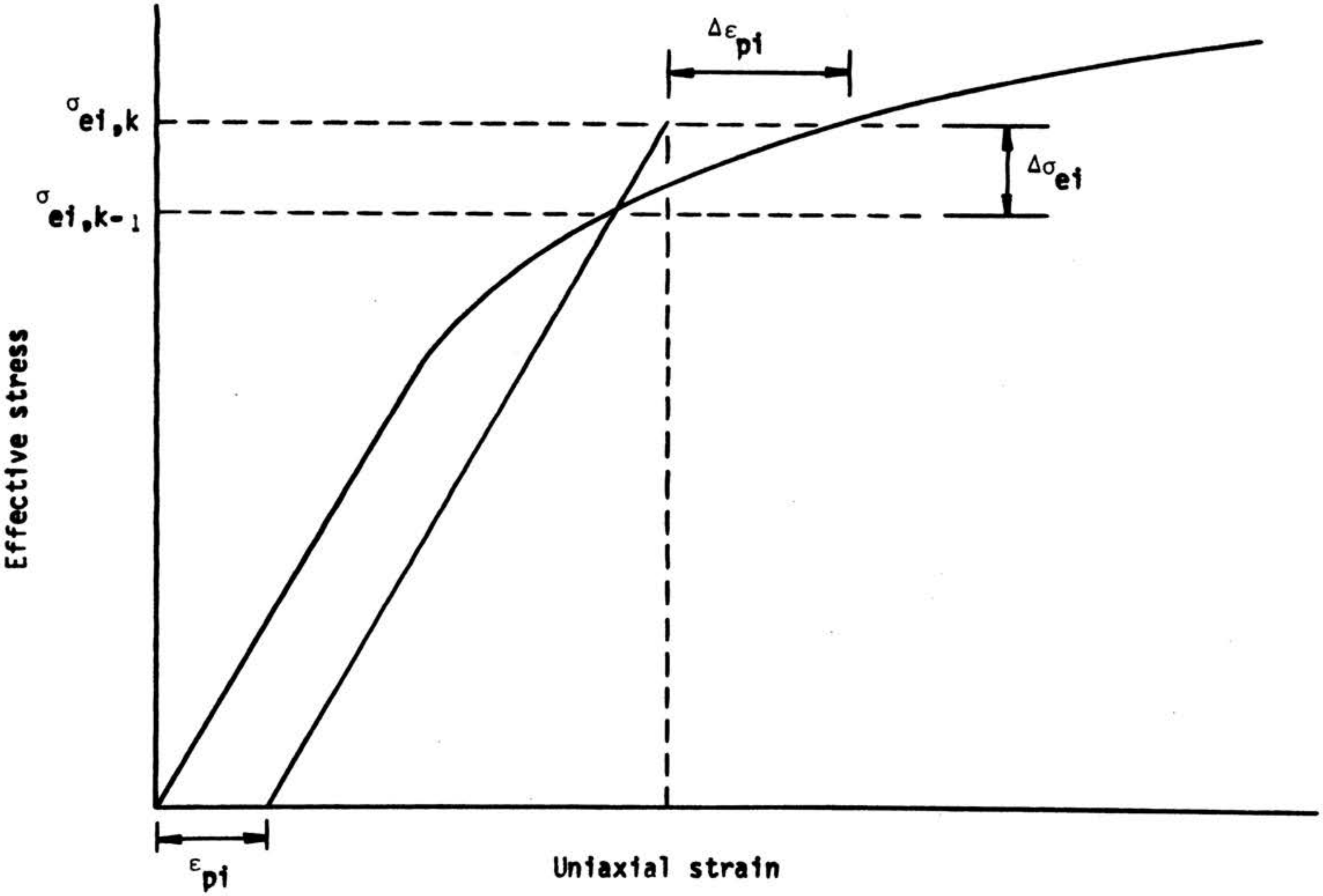

Figure 2. Relation Between, $\Delta\varepsilon_p$, ε_p and σ_e

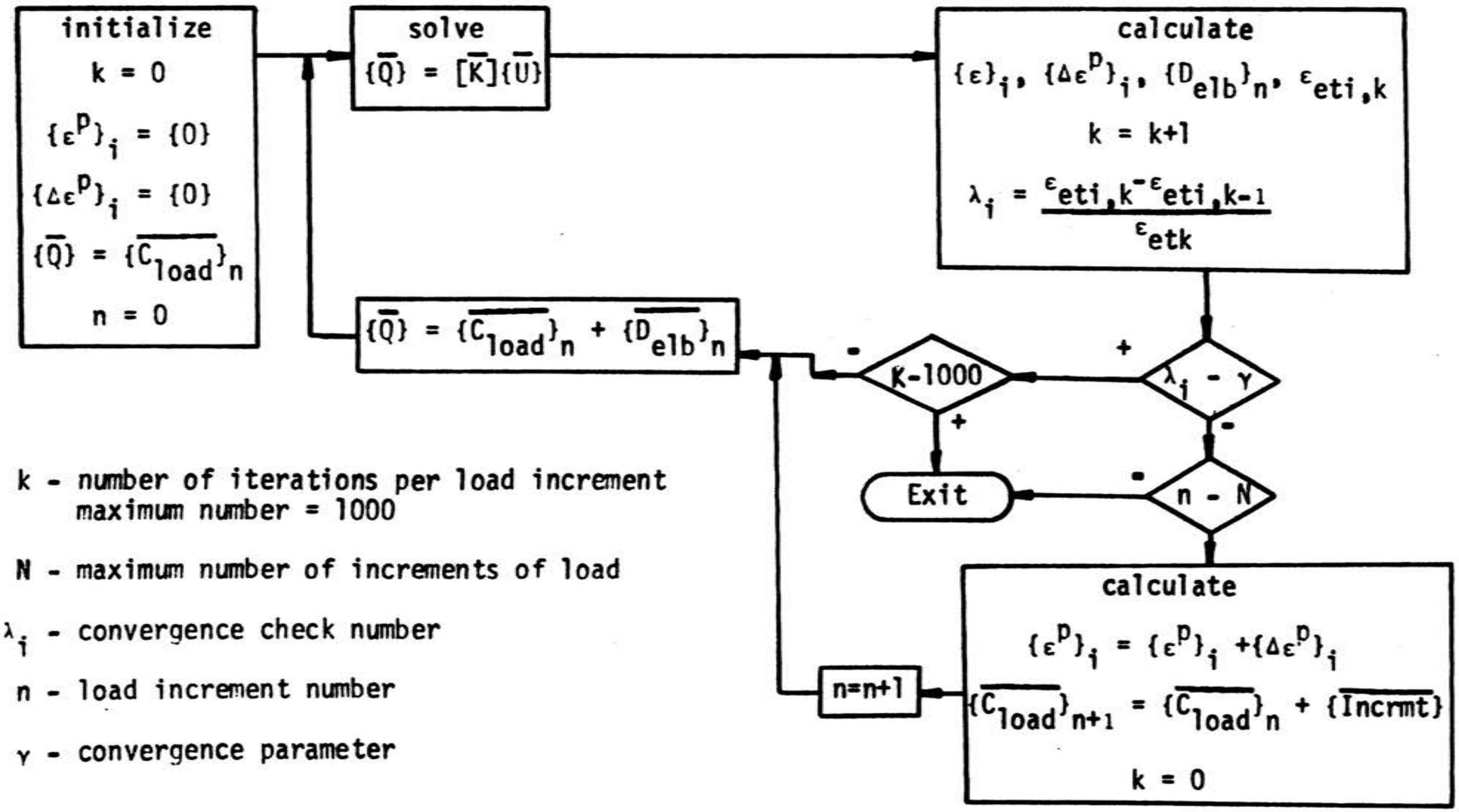

Figure 3. Flow Chart of Successive Elastic Solutions

Yield stress: 40,000 psi

Young's Modulus: 30×10^6 psi

Plastic modulus: $.069 \times 10^6$ psi

Poisson's Ratio: .3

Applied stress: 45,000 psi

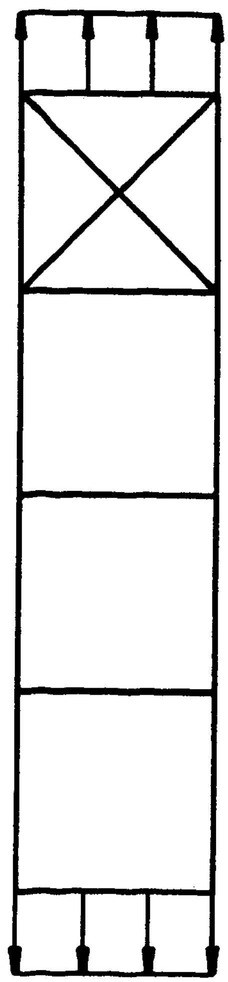

Figure 4. Test Problem, Tension Strip

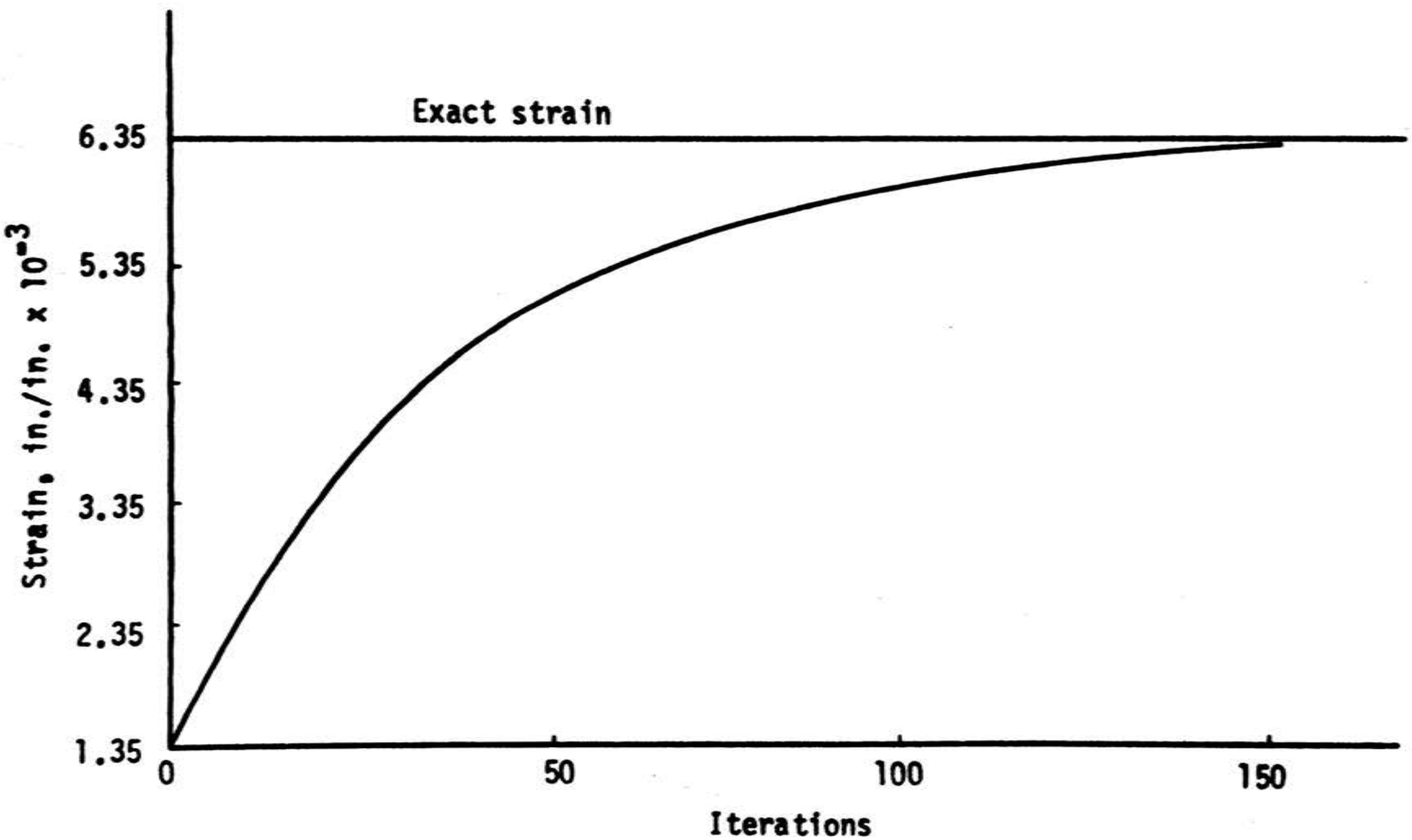

Figure 5. Iterations vs. Strain for Tension Strip

Material: Aluminum alloy 57S
Yield stress: 24.3 Kg mm^{-2}
Young's Modulus: 7000 Kg mm^{-2}
Plastic Modulus: 220 Kg mm^{-2}

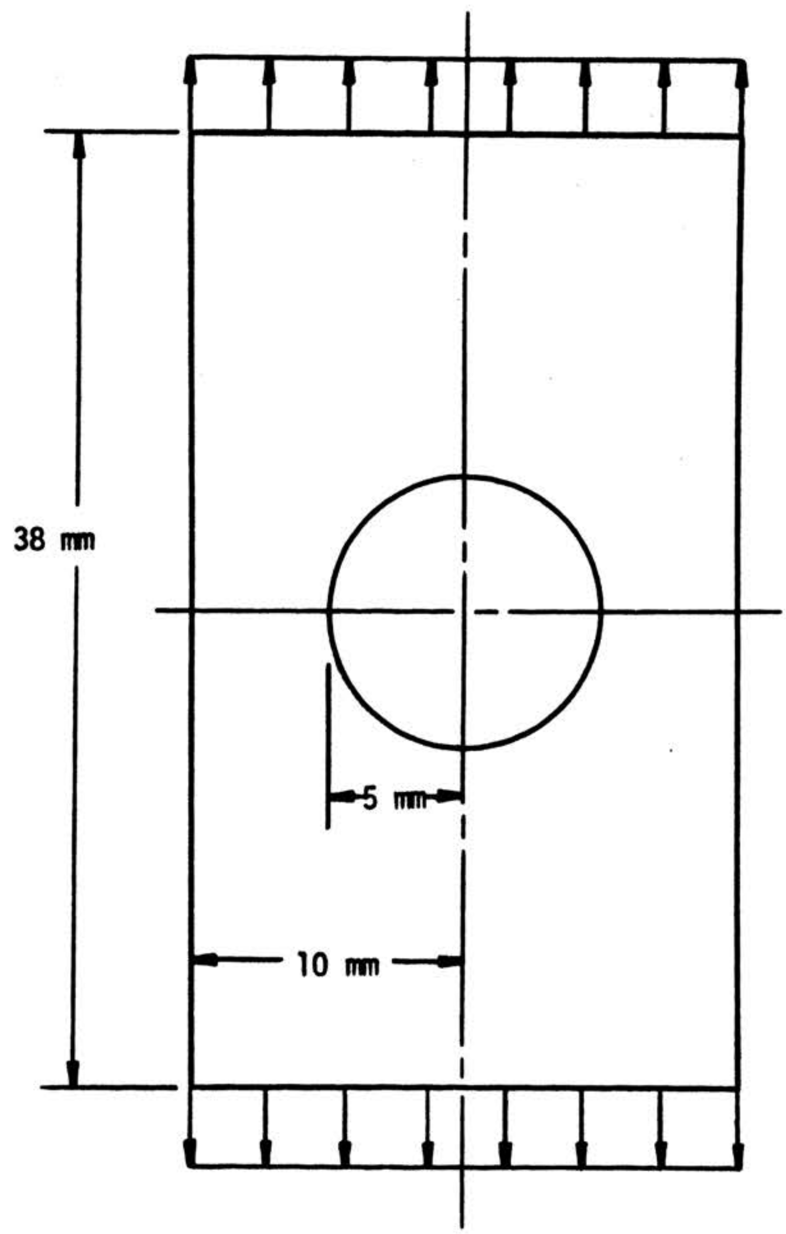

Figure 6. Perforated Tension Strip

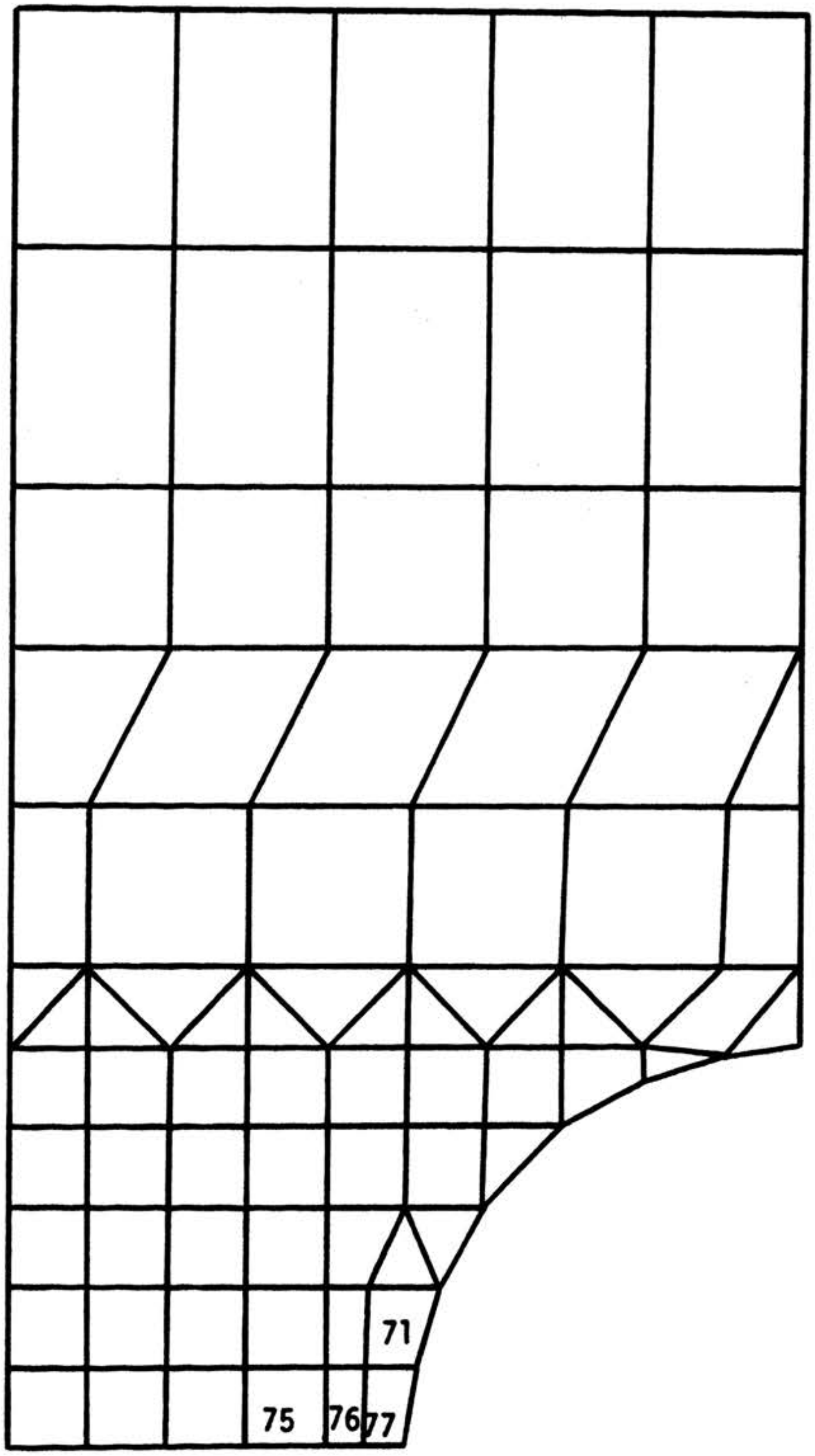

Figure 7. Test Problem 2, Perforated Tension Strip

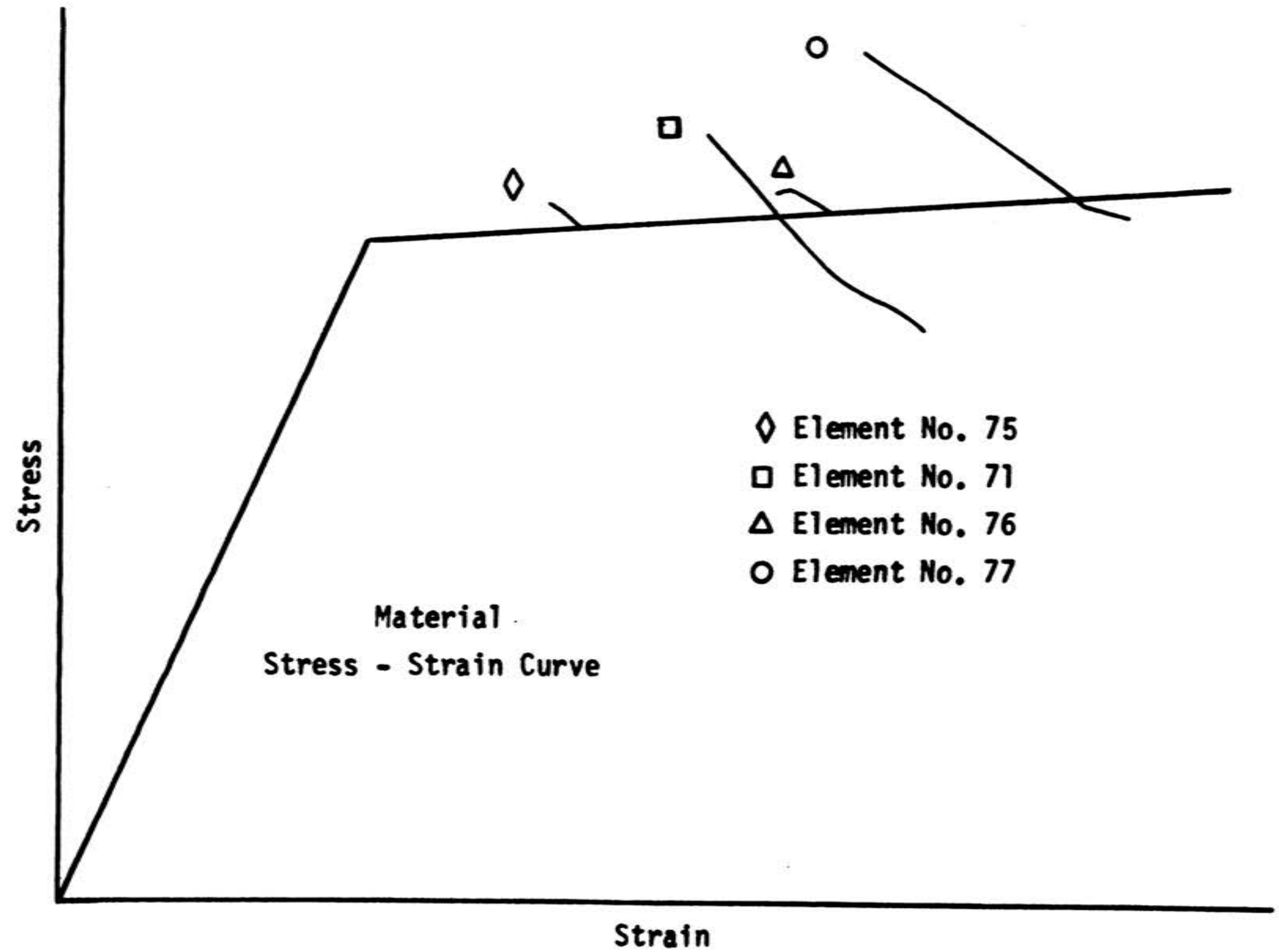

Figure 8. Path of Element Stress and Strain During Iteration

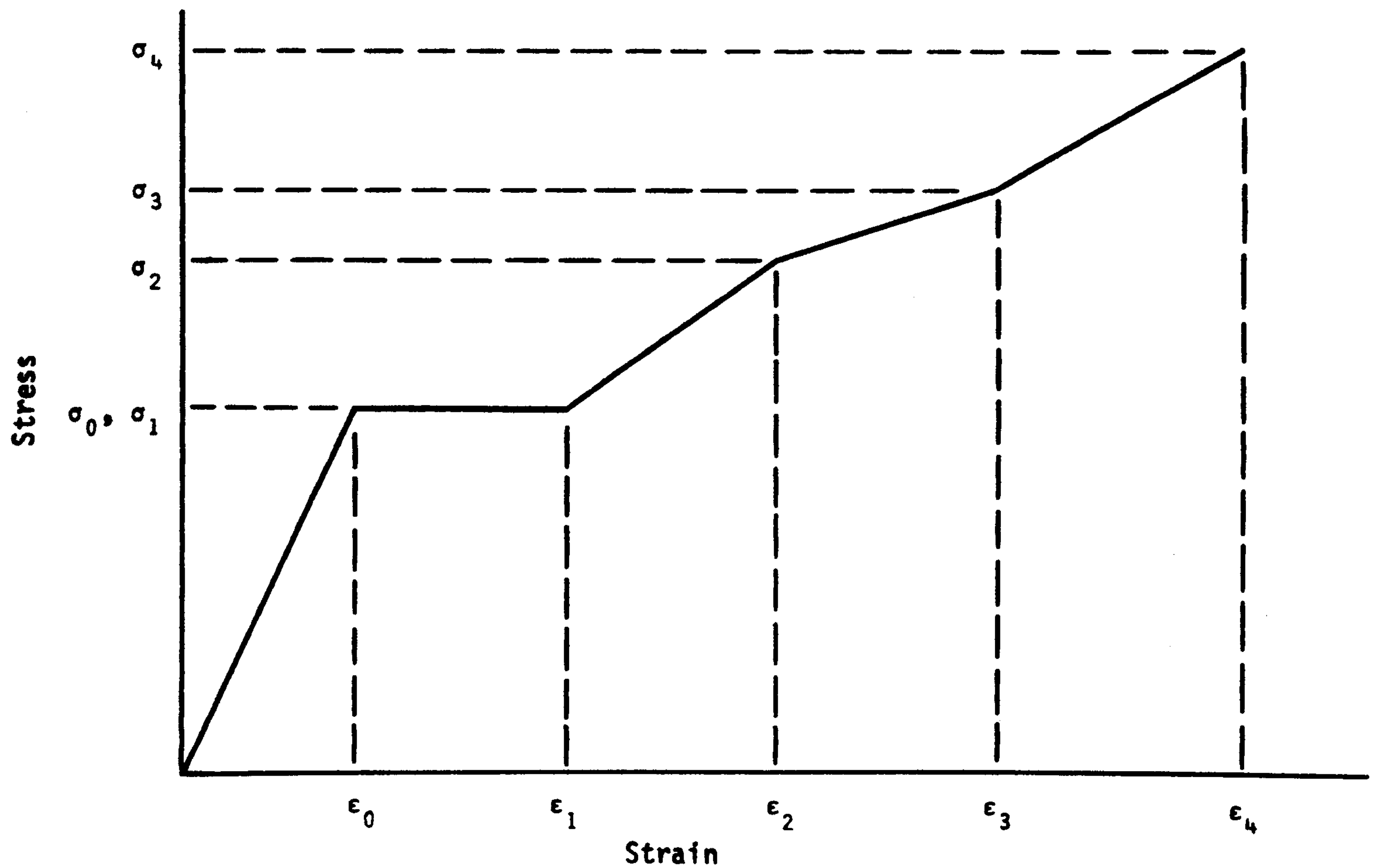

Figure 9. Material Uniaxial Stress - Strain Curve

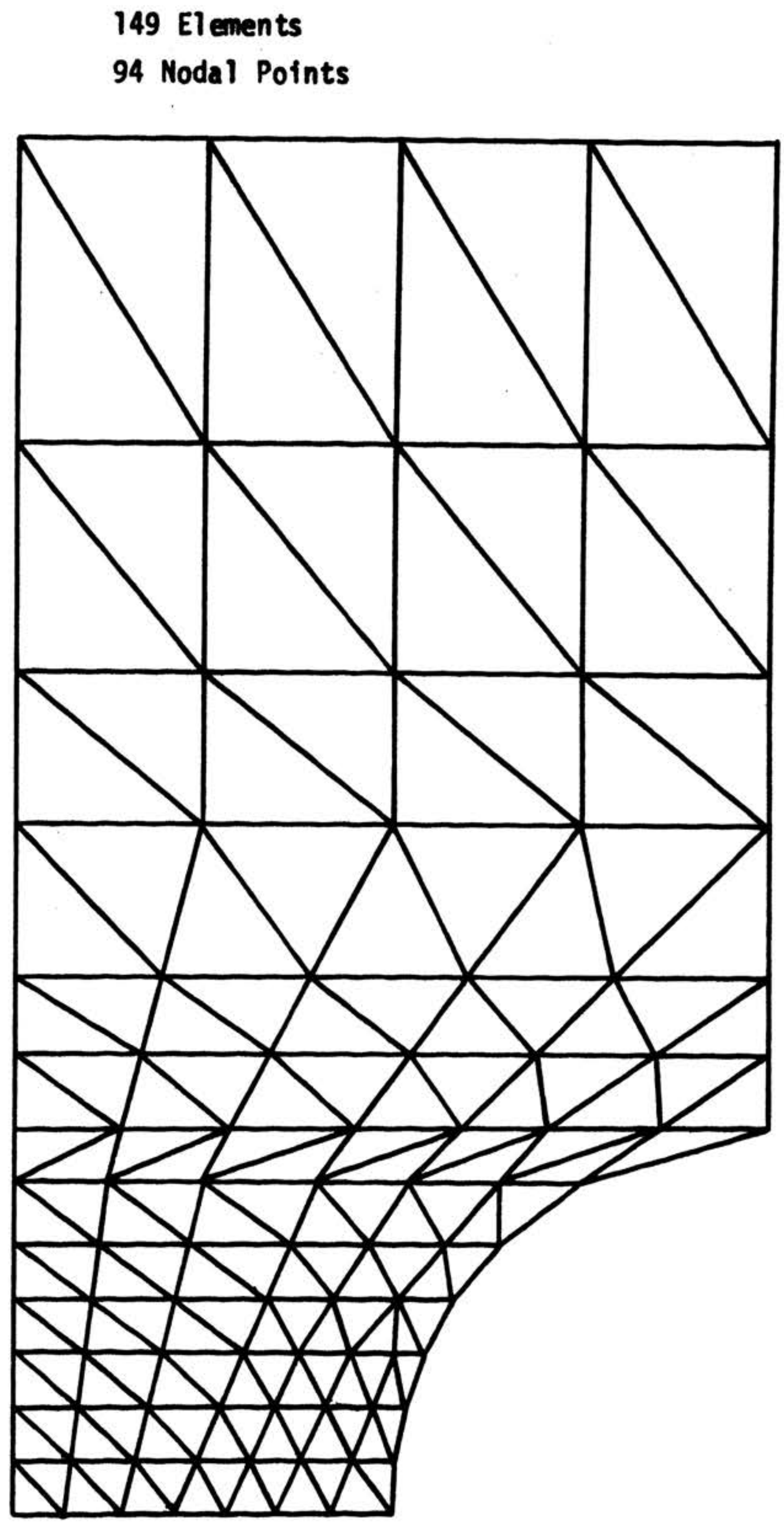

Figure 10. Perforated Tension Strip - Element Grid

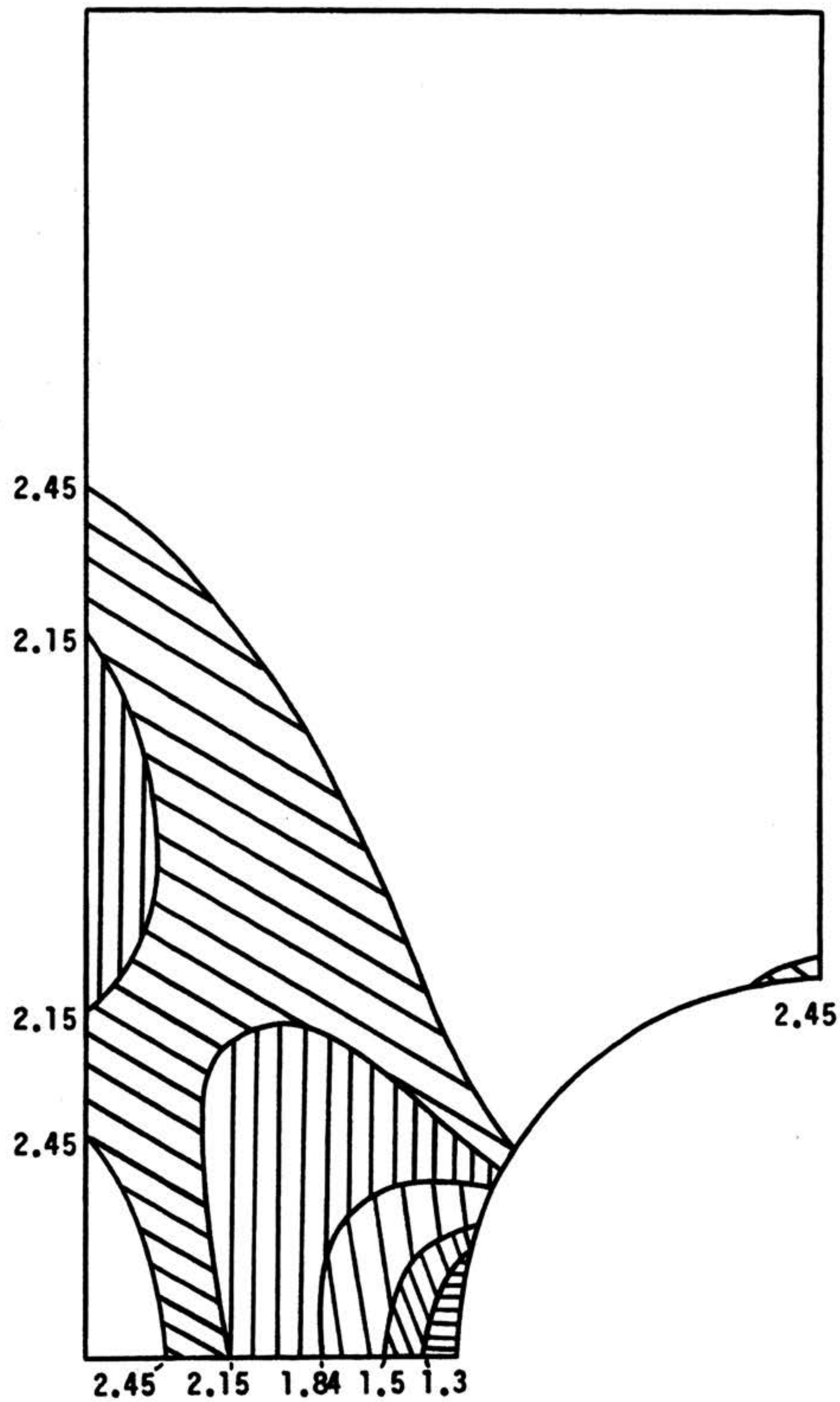

Figure 11. Perforated Tension Strip (Experimental)

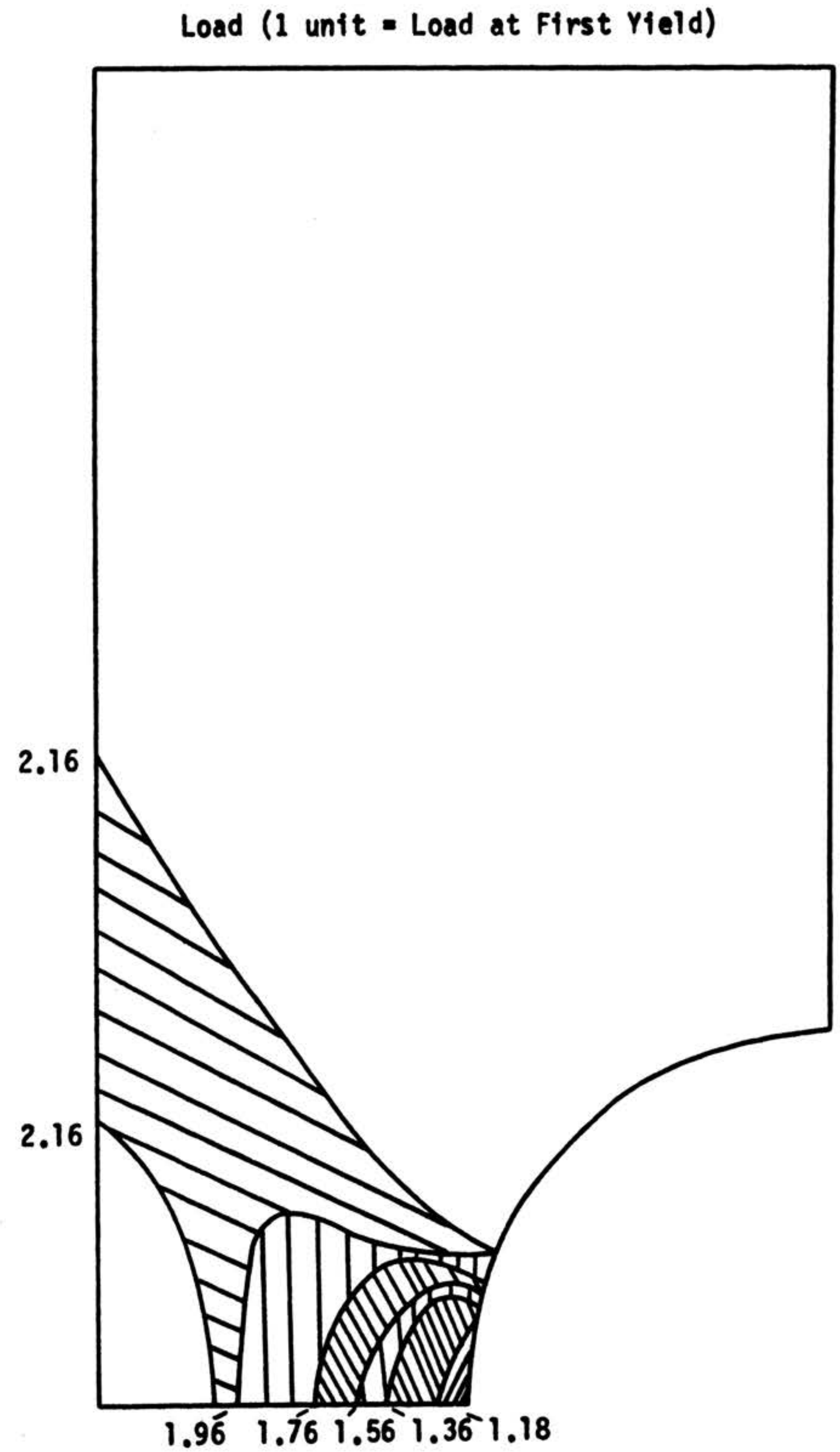

Figure 12. Perforated Tension Strip (Initial Stress Method)

Load (1 unit = Load at First Yield)

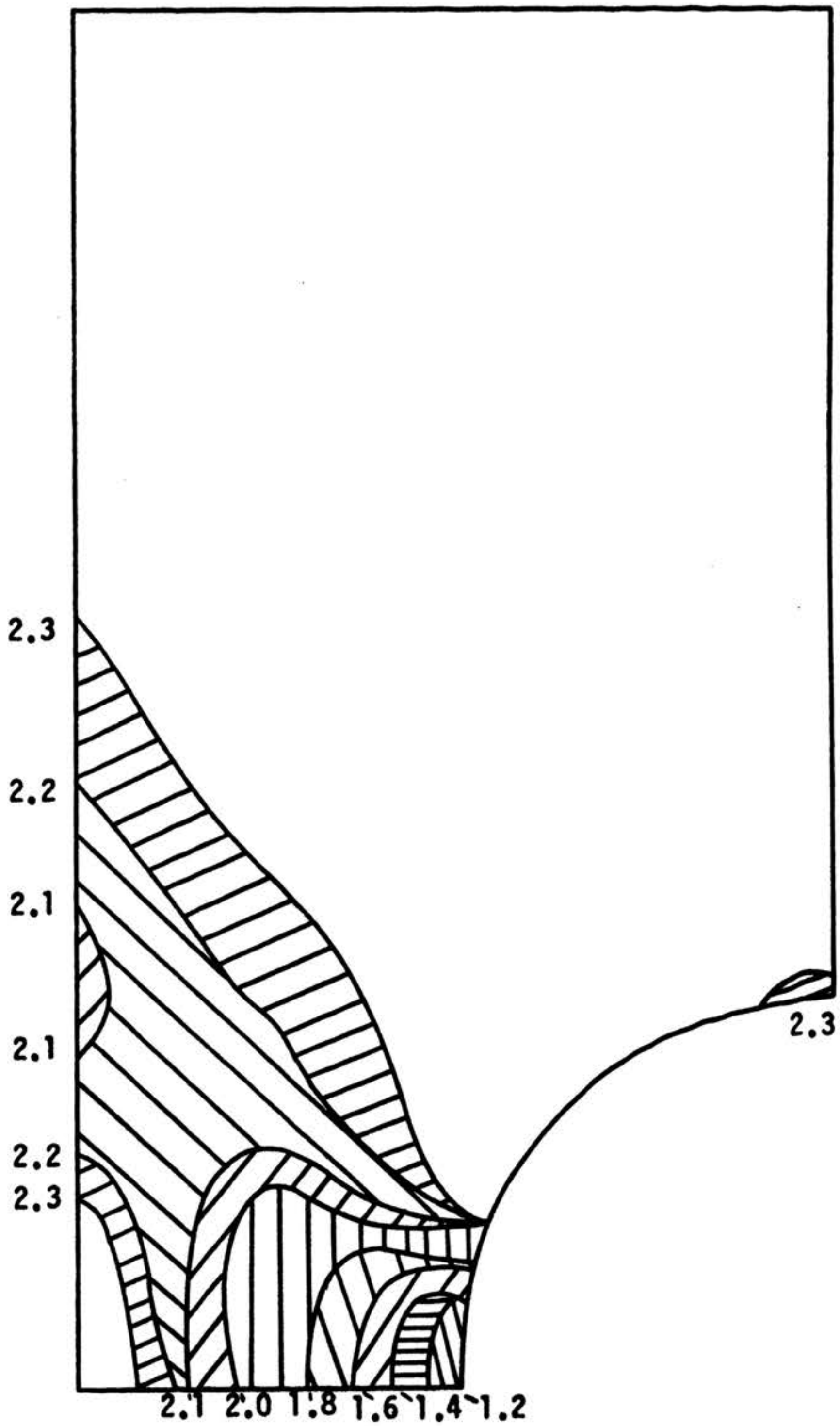

Figure 13. Perforated Tension Strip (Partial Stiffness Method)

Load (1 unit = Load at First Yield)

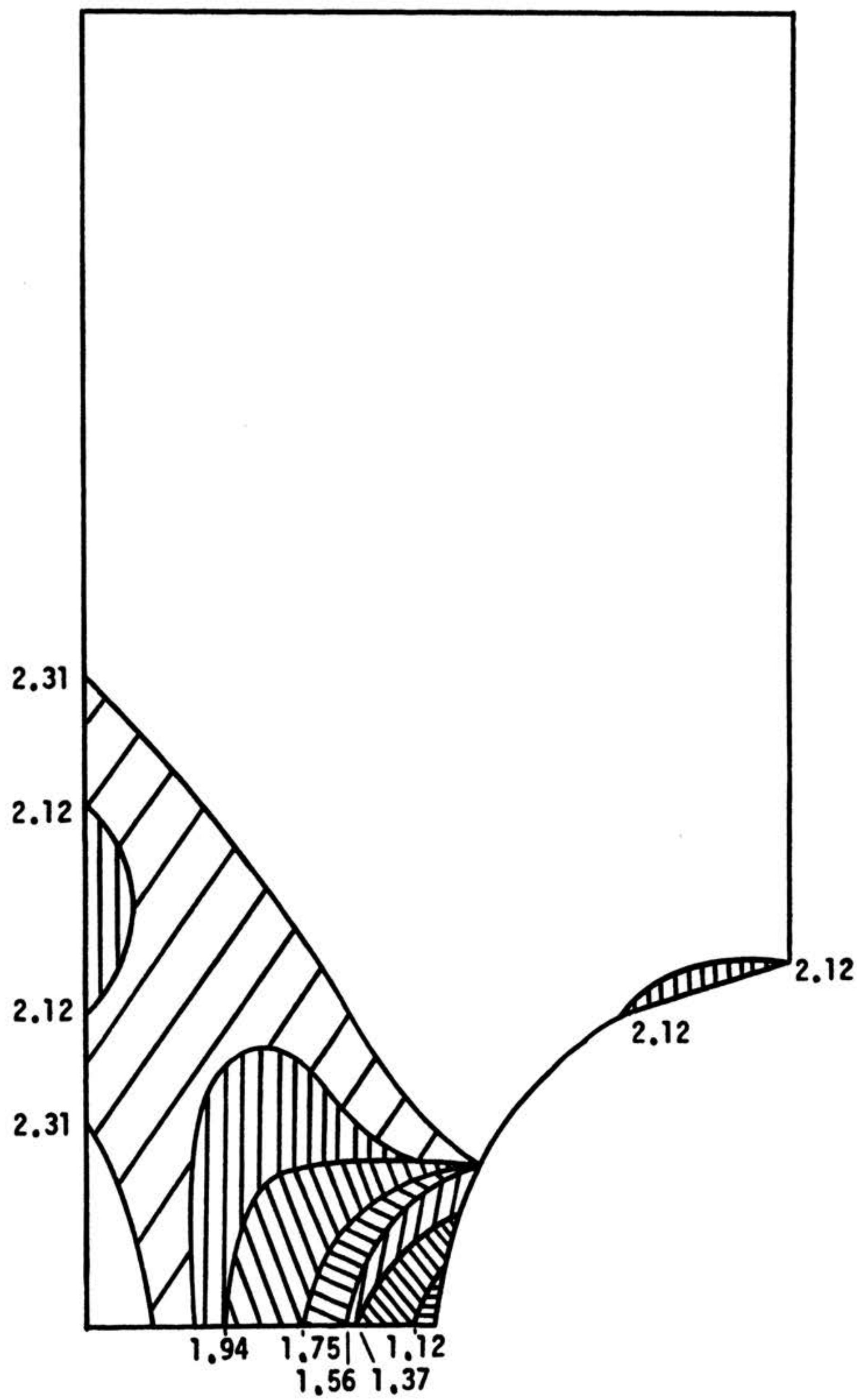

Figure 14. Perforated Tension Strip (Initial Strain Approach)

Material: Aluminum alloy
Yield stress: 24.3 Kg mm^{-2}
Young's Modulus: 7000 Kg mm^{-2}
Ideal plasticity

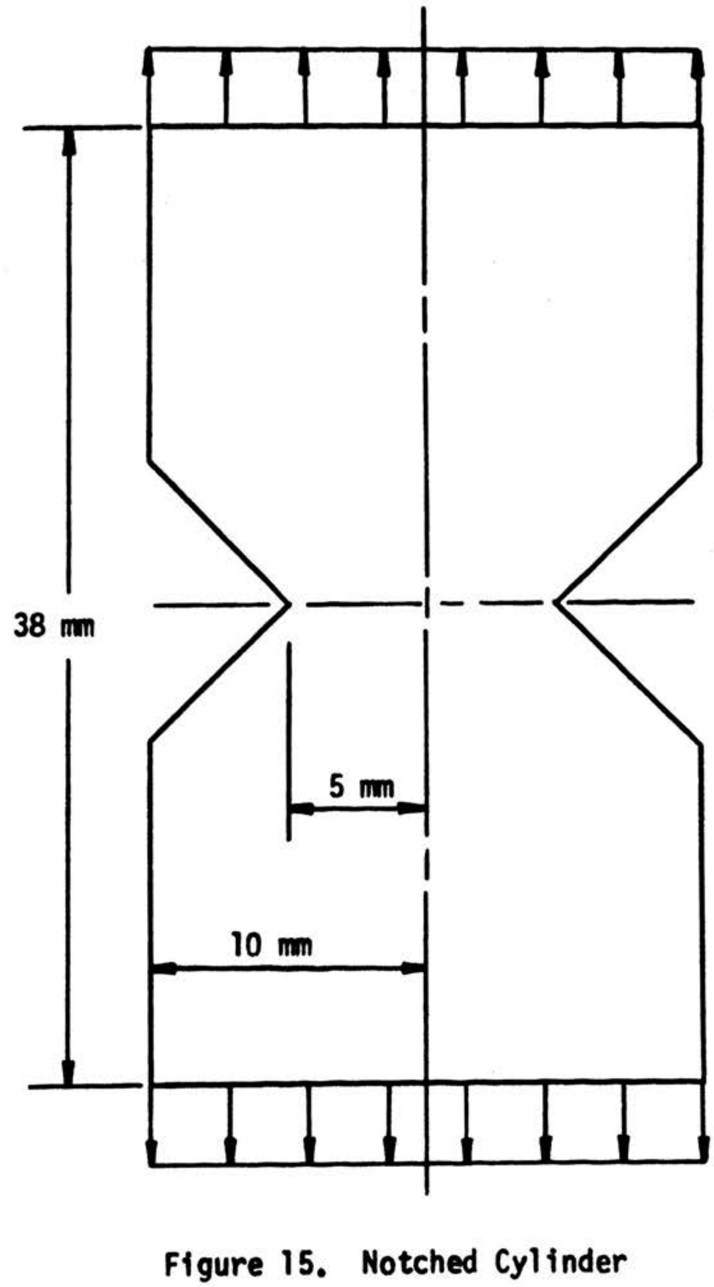

Figure 15. Notched Cylinder

Axisymmetric cylinder
107 Elements
119 Nodal points
Ideal plasticity

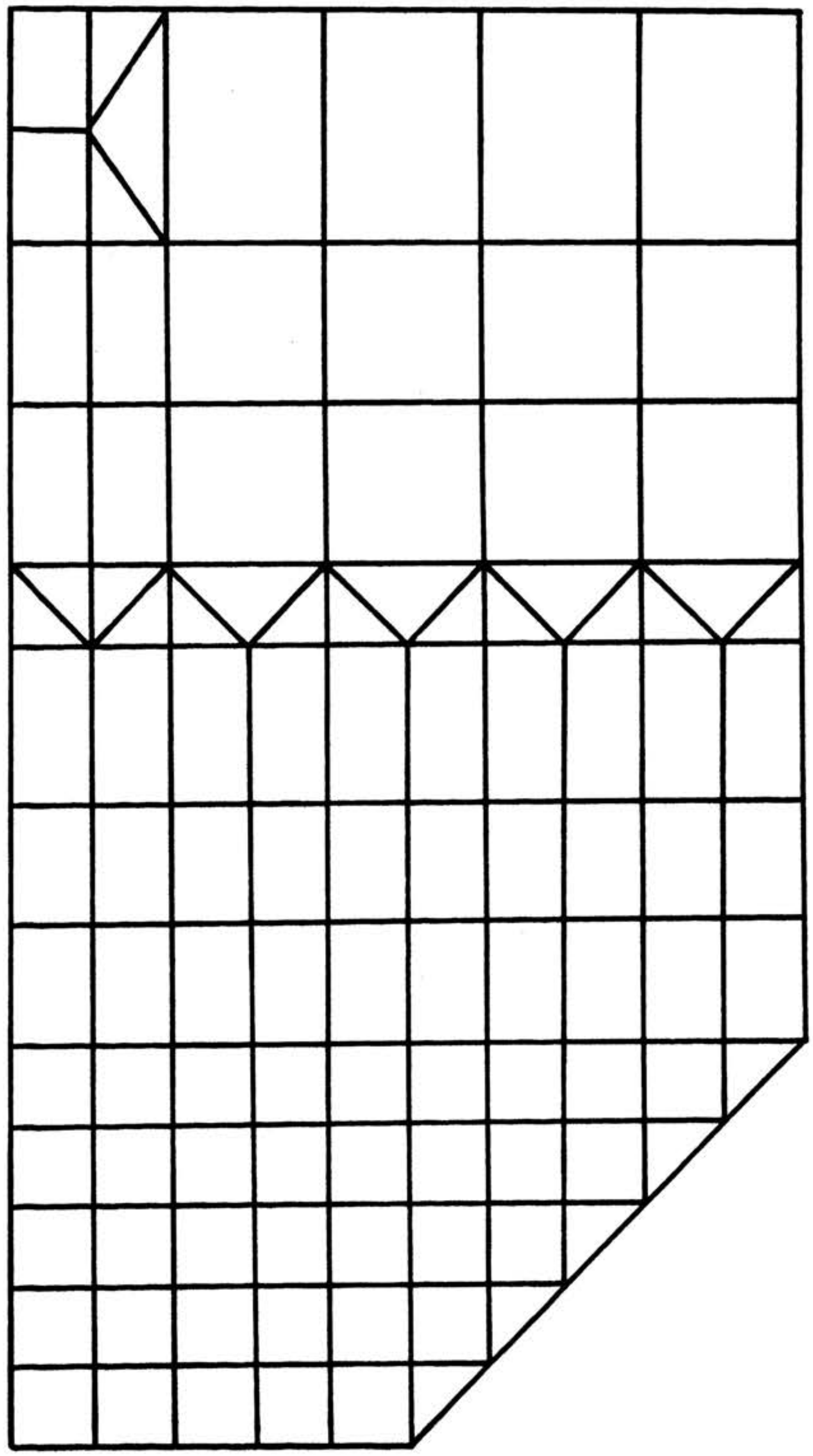

Figure 16. Element Grid

Load (1 Unit = Load at First Yield)

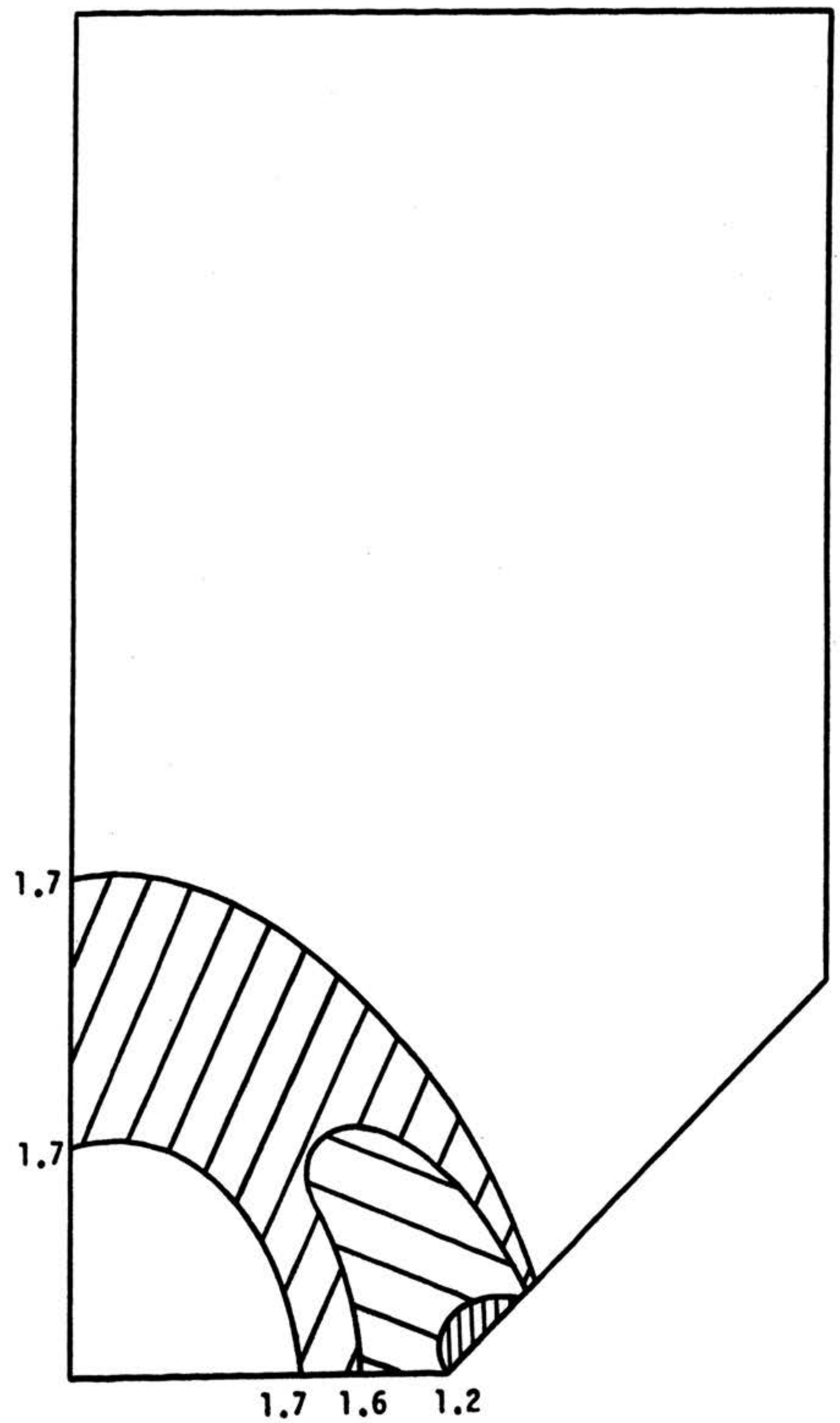

Figure 17. Plastic Enclaves - Notched Cylinder

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VITA

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APPENDIX A

SOLUTION OF EQUATIONS

Once the mechanical and thermal loads are calculated the solution for the nodal point displacements can be found by solving

$$\{\overline{Q}\} = [\overline{K}]\{\overline{U}\} \tag{A-1}$$

If Gaussian elimination is used a matrix equation results

$$\{\overline{Q'}\} = [\overline{K'}]\{\overline{U}\} \tag{A-2}$$

where

$[\overline{K'}]$ is an upper triangle matrix.

The reduction of $\{\overline{Q}\}$ to $\{\overline{Q'}\}$ is accomplished with the diagonal elements of the original $[\overline{K}]$ matrix. Any new force vector can be reduced as was $\{\overline{Q}\}$. Equation (A-2) is solved by back substitution for the nodal point displacements. To conserve storage only the upper triangular half of the $[\overline{K}]$ matrix is stored. This upper triangular matrix is shifted so the diagonal elements occupy the first column in the storage matrix. When $[\overline{K'}]$ is formed the original diagonal elements of $[\overline{K}]$ are stored in the first column of the storage area reserved for the shifted upper triangular $[\overline{K'}]$.

To solve for displacements after plastic loads are applied, the new load matrix is reduced and the back substitution is performed. It is necessary to reduce the $[\overline{K}]$ only once.

APPENDIX B

MODIFICATIONS TO WILSON'S PROGRAM

This appendix gives a brief discussion of how the Wilson program handles quadrilateral elements and then how quadrilateral elements were treated for the elastic and plastic cases during this study. This discussion will include calculation of element plastic loads. Subroutines from the Wilson program and the new program will be referred to by name.

A. The Wilson Program

The Wilson program assumed a linear displacement field for each element which results in constant strain in each element. The basic element was a triangle from which quadrilateral elements could be formed. The quadrilateral is divided into four triangles by placing an additional nodal point at the centroid of the element. The stiffness of each of four triangular elements is calculated and combined to make the stiffness of the quadrilateral element. The addition of the fifth nodal point increases the size of the element stiffness matrix from an (8 x 8 matrix) to a (10 x 10 matrix), and the displacement vector becomes a (10 x 1 vector). The fifth nodal point is eliminated from the element stiffness matrix leaving the stiffness in terms of the four corner nodal points.

After the displacements are obtained for the structure from equation (2.11), a set of displacements for the fifth nodal point of the quadrilateral is calculated. The ten displacements are premultiplied by $[g]_i[H]_i$ to find the element strain.

$$\{\varepsilon\}_i = [g]_i[H]_i\{u\}_i \tag{B-1}$$

where

$[g]_i$ - is a function of location in the element and is a (4 x 6) matrix

$[H]_i$ - is independent of location in the element and is a (6 x 10) matrix

$\{u\}_i$ - element nodal point displacement for four corner nodal points plus centroidal nodal point

$$[a]_i = [g]_i[H]_i \tag{B-2}$$

B. Modification of the Wilson Stress Subroutine

A convenient method of determining stresses and strains from the four nodal point displacements for quadrilateral elements was needed. The purely elastic case was considered first and then modified for the case when plastic strains occurred.

The equation relating nodal point displacements and nodal point loads for the elastic case is

$$\left\{\begin{array}{c} q(1) \\ \vdots \\ \hline q(9) \\ q(10) \end{array}\right\}_i = \left[\begin{array}{ccc|cc} k(1,1) & \cdots & k(1,8) & k(1,9) & k(1,10) \\ \vdots & & \vdots & \vdots & \vdots \\ \hline k(9,1) & \cdots & k(9,8) & k(9,9) & k(9,10) \\ k(10,1) & \cdots & k(10,8) & k(10,9) & k(10,10) \end{array}\right]_i \left\{\begin{array}{c} u(1) \\ \vdots \\ \hline u(9) \\ u(10) \end{array}\right\}_i \tag{B-3}$$

where

q(j) - nodal point loads, from mechanical and thermal loads

u(j) - nodal point displacements

k(j,j) - element stiffness matrix

j - index of nodal point displacements, for quadrilateral j varies from 1 to 10, for triangles j varies from 1 to 6

The matrix equation (B-3) can be partitioned as shown and rewritten as

$$\begin{Bmatrix} \tilde{q}1 \\ \hline \tilde{q}2 \end{Bmatrix}_i = \left[\begin{array}{c|c} \tilde{k}1 & \tilde{k}2 \\ \hline \tilde{k}3 & \tilde{k}4 \end{array}\right]_i \begin{Bmatrix} \tilde{u}1 \\ \hline \tilde{u}2 \end{Bmatrix}_i \tag{B-4}$$

or

$$\begin{aligned} \{q1\}_i &= [k1]_i\{u1\}_i + [k2]_i\{u2\}_i \\ \{q2\}_i &= [k3]_i\{u1\}_i + [k4]_i\{u2\}_i \end{aligned} \tag{B-5}$$

where

$\{q1\}_i$ - (8 x 1) vector of loads

$\{q2\}_i$ - (2 x 1) vector of loads

$[k1]_i$ - (8 x 8) stiffness matrix

$[k2]_i$ - (8 x 2) stiffness matrix

$[k3]_i$ - (2 x 8) stiffness matrix

$[k4]_i$ - (2 x 2) stiffness matrix

$\{u1\}_i$ - (8 x 1) vector of nodal point displacements

$\{u2\}_i$ - (2 x 1) vector of nodal point displacements

Solving equation (B-5) for the last two nodal point displacements $\{u2\}_i$ gives

$$\{u2\}_i = [k4]_i^{-1}\{q2\}_i - [k4]_i^{-1}[k3]_i\{u1\}_i \tag{B-6}$$

Recalling the equation for strain

$$\{\varepsilon\}_i = [g]_i[H]_i\{u\}_i \tag{B-7}$$

Let

$$\{\alpha\}_i = [H]_i\{u\}_i \tag{B-8}$$

or in expanded form

$$\left\{\begin{array}{c} \alpha(1) \\ \alpha(2) \\ \alpha(3) \\ \alpha(4) \\ \hline \alpha(5) \\ \alpha(6) \end{array}\right\}_i = \left[\begin{array}{cccc|cc} H(1,1) & \cdot & \cdot\;\cdot\;\cdot & H(1,8) & H(1,9) & H(1,10) \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \hline H(5,1) & \cdot & \cdot\;\cdot\;\cdot & H(5,8) & H(5,9) & H(5,10) \\ H(6,1) & \cdot & \cdot\;\cdot\;\cdot & H(6,8) & H(6,9) & H(6,10) \end{array}\right]_i \left\{\begin{array}{c} u(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hline u(9) \\ u(10) \end{array}\right\}_i \tag{B-9}$$

Writing $\{\alpha\}_i$ as

$$\left\{\begin{array}{c} \tilde{\alpha}1 \\ \hline \tilde{\alpha}2 \end{array}\right\}_i = \left[\begin{array}{c|c} \tilde{H}1 & \tilde{H}2 \\ \hline \tilde{H}3 & \tilde{H}4 \end{array}\right]_i \left\{\begin{array}{c} \tilde{u}1 \\ \hline \tilde{u}2 \end{array}\right\}_i \tag{B-10}$$

then

$$\begin{aligned} \{\alpha 1\}_i &= [H1]_i\{u1\}_i + [H2]_i\{u2\}_i \\ \{\alpha 2\}_i &= [H3]_i\{u1\}_i + [H4]_i\{u2\}_i \end{aligned} \tag{B-11}$$

Recalling the equation for $\{u2\}_i$ previously developed, equations (B-11) become

$$\begin{aligned} \{\alpha 1\}_i &= [H1]_i\{u1\}_i+[H2]_i([k4]^{-1}\{q2\}-[k4]^{-1}[k3]\{u1\})_i \\ \{\alpha 2\}_i &= [H3]_i\{u1\}_i\ [H4]_i([k4]^{-1}\{q2\}-[k4]^{-1}[k3]\{u1\})_i \end{aligned} \tag{B-12}$$

Combining like terms

$$\{\alpha 1\}_i = [\tilde{H}1 - \tilde{H}2\ \tilde{k}4^{-1}\ \tilde{k}3]_i\{u1\}_i + [H2]_i[k4]_i^{-1}\{q2\}_i$$
$$\{\alpha 2\}_i = [\tilde{H}3 - \tilde{H}4\ \tilde{k}4^{-1}\ \tilde{k}3]_i\{u1\}_i + [H4]_i[k4]_i^{-1}\{q2\}_i \tag{B-13}$$

$$\{\alpha\}_i = \begin{bmatrix} \tilde{H}1 - \tilde{H}2\ \tilde{k}4^{-1}\ \tilde{k}3 \\ \tilde{H}3 - \tilde{H}4\ \tilde{k}4^{-1}\ \tilde{k}3 \end{bmatrix}_i \{u1\}_i + \begin{bmatrix} \tilde{H}2\ \tilde{k}4^{-1} \\ \tilde{H}4\ \tilde{k}4^{-1} \end{bmatrix}_i \{q2\}_i \tag{B-14}$$

In the subroutine quad

$$[AH]_i = [k4]_i^{-1} \tag{B-15}$$

$$[F_a]_i = \begin{bmatrix} \tilde{H}2\ \tilde{k}4^{-1} \\ \tilde{H}4\ \tilde{k}4^{-1} \end{bmatrix}_i \tag{B-16}$$

$$[S]_i = [k3]_i \tag{B-17}$$

$$[HI]_i = \begin{bmatrix} \tilde{H}1 - \tilde{H}2\ \tilde{k}4^{-1}\tilde{k}3 \\ \tilde{H}3 - \tilde{H}4\ \tilde{k}4^{-1}\tilde{k}3 \end{bmatrix}_i \tag{B-18}$$

The equation for $\{\alpha\}_i$ becomes

$$\{\alpha\}_i = [HI]_i\{u1\}_i + [F_a]_i\{q2\}_i \tag{B-19}$$

For the elastic problem $\{q2\}_i$ will be the last two nodal loads, and the produce $[F_a]_i\{q2\}_i$ will be a constant.

The strain can now be written as

$$\{\varepsilon\}_i = [g]_i[HI]_i\{u1\}_i + [g]_i[F_a]_i\{q2\}_i\ . \tag{B-20}$$

Let

$$[HH]_i = [g]_i[HI]_i \tag{B-21}$$

$$\{AI\}_i = [g]_i[F_a]_i\{q2\}_i \tag{B-22}$$

so

$$\{\varepsilon\}_i = [HH]_i\{u1\}_i + \{AI\}_i \tag{B-23}$$

which represents the relationship between the first eight nodal displacements and the elastic strain of an element.

For the case of plastic strain recall equation (B-19) for $\{\alpha\}_i$ and modify $\{u1\}_i$ and $\{q2\}_i$ to account for plastic strain at the fifth nodal point.

$$\{\alpha\}_i = [HI]_i\{u1\}_i + [F_a]_i\{\tilde{q}2 + \tilde{b}2\}_i \tag{B-24}$$

$$\{\alpha\}_i = [HI]_i\{u1\}_i + [F_a]_i\{q2\}_i + [F_a]_i\{\tilde{b}2\}_i \tag{B-25}$$

where

$\{b2\}_i$ - plastic loads at fifth nodal point due to plastic strains in the four triangles of the quadrilateral

Premultiplying by $[g]_i$

$$\{\varepsilon\}_i = [HH]_i\{u1\}_i + \{AI\}_i + [g]_i[F_a]_i\{b2\}_i \tag{B-26}$$

The equation for plastic loads is developed in section C of Appendix B and is

$$\{b\}_i = [FH]_i\{\varepsilon^{pt}\}_i \tag{B-27}$$

where

$\{b\}_i$ - element load vector due to element plastic strain

$\{\varepsilon^{pt}\}_i$ - element total plastic strain vector

Expanding equation (B-27) gives

$$\begin{Bmatrix} b(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hline b(9) \\ b(10) \end{Bmatrix}_i = \begin{bmatrix} FH(1,1) & \cdots & FH(1,4) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \hline FH(9,1) & \cdots & FH(9,4) \\ FH(10,1) & \cdots & FH(10,4) \end{bmatrix}_i \{\varepsilon^{pt}\}_i \tag{B-28}$$

or

$$\begin{Bmatrix} \tilde{b}1 \\ \hline \tilde{b}2 \end{Bmatrix}_i = \begin{bmatrix} \tilde{F}H1 \\ \hline \tilde{F}H2 \end{bmatrix}_i \{\varepsilon^{pt}\}_i \tag{B-29}$$

and

$$\{b2\}_i = [FH2]_i\{\varepsilon^{pt}\}_i \tag{B-30}$$

The plastic loads do not exist unless plastic strains have occurred. The matrix $[FH2]_i$ is calculated and stored in the subroutine for calculating plastic strains. The strain then becomes

$$\{\varepsilon\}_i = [HH]_i\{u1\}_i + \{AI\}_i + [g]_i[F_a]_i[FH2]_i\{\varepsilon^{pt}\}_i \tag{B-31}$$

Let

$$[EH]_i = [FH2]_i \tag{B-32}$$

$$[A1P]_i = [g]_i[F_a]_i[FH2]_i \tag{B-33}$$

Equation (B-31) can then be written as

$$\{\varepsilon\}_i = [HH]_i\{u1\}_i + \{AI\}_i + [A1P]_i\{\varepsilon^{pt}\}_i \tag{B-34}$$

C. Additional Subroutine Delep to Convert Plastic Strain to Element Loads

A new subroutine, Delep, was added to the program to calculate plastic loads. For the case of a triangular element the loads are placed on three nodes and for the quadrilateral element the loads are placed on the four corners nodes.

The equation relating loads to plastic strains and body forces, equation (2.9) is

$$\{L\}_i = \int_{Vol\ i} ([d]_i^T\{F\}_i + [a]_i^T[c]_i\{\varepsilon^{pt}\}_i)dV \tag{B-35}$$

or for only plastic loads,

$$\{b\}_i = \int_{Vol\ i} [a]_i^T[c]_i\{\varepsilon^{pt}\}_i dV \tag{B-36}$$

Substituting from equation (B-2) for $[a]_i^T$

$$\{b\}_i = \int_{Vol\ i} [H]_i^T[g]_i^T[c]_i\{\varepsilon^{pt}\}_i dV \tag{B-37}$$

or

$$\{b\}_i = [H]_i^T \int_{Vol\ i} [g]_i^T[c]_i\{\varepsilon^{pt}\}_i dV \tag{B-38}$$

For a triangular element $[H]_i^T$ and $\int_{Vol\ i} [g]_i^T[c]_i\{\varepsilon^{pt}\}_i dV$ depend only on the particular triangular element chosen. A quadrilateral is made of four triangles, $[H]_i^T$ and $\int_{Vol\ i} [g]_i^T[c]_i\{\epsilon^{pt}\}_i dV$ are different for each triangle.

The equation for plastic loads on any triangle can be written as

$$\{b\}_i = [H]_i^T \int_{Vol\ m} [g]_i^T dV_i [c]_i \{\epsilon^{pt}\}_i \tag{B-39}$$

The Wilson program uses constant strain elements which allows $\{\epsilon^{pt}\}_i$ to be moved outside the integration in equation (B-39). In the quadrilateral each of the four triangles are treated separately and then combined.

$$\{b\}_i = \{b^1\}_i + \{b^2\}_i + \{b^3\}_i + \{b^4\}_i \tag{B-40}$$

$$\begin{aligned}\{b\}_i = {} & [H^1]^T \int_{V_1} [g^1]^T dV_1 [c]_i \{\epsilon^{pt}\}_i + [H^2]^T \int_{V_2} [g^2]^T dV_2 [c]_i \{\epsilon^{pt}\}_i \\ & + [H^3]^T \int_{V_3} [g^3]^T dV_3 [c]_i \{\epsilon^{pt}\}_i + [H^4]^T \int_{V_4} [g^4]^T dV_4 [c]_i \{\epsilon^{pt}\}_i\end{aligned} \tag{B-41}$$

or

$$\{b\}_i = \sum_{m=1}^{4} [H^m]_i \int_{V_m} [g^m]_i^T dV_m [c]_i \{\epsilon^{pt}\}_i \tag{B-42}$$

where

m - indicates one of the four triangles which form the quadrilateral

$\{\epsilon^{pt}\}_i$ - is the plastic strain for the quadrilateral

$[c]_i$ - is the elasticity matrix

The matrices are expanded to allow addition of the matrices

$\{b\}_i$ from a (6 x 1) to a (10 x 1 vector)

$[H]_i$ from a (6 x 6) to a (6 x 10 matrix)

The integrals are evaluated

$$[D^m]_i = \int_{V_m} [g^m]_i^T dV_m [c]_i \tag{B-43}$$

$[D^m]_i$ - (6 x 4) matrix

and then summed,

$$\{b\}_i = \sum_{m=1}^{4} [H^m]_i^T [D^m]_i \{\varepsilon^{pt}\}_i \tag{B-44}$$

Let

$$[FH]_i = \sum_{m=1}^{4} [H^m]_i^T [D^m]_i \tag{B-45}$$

then

$$\{b\}_i = [FH]_i \{\varepsilon^{pt}\}_i \tag{B-46}$$

If the element is a triangle the calculation is stopped after m = 1 is calculated in equation (B-45) and $[FH]_i$ is stored on magnetic disk. For the quadrilateral element the complete $[FH]_i$ is calculated prior to storing on magnetic disk.

The load matrix calculated from equation (B-46) for a quadrilateral will have loads for five nodal points. The fifth node of the quadrilateral elements was eliminated for the elastic solution and must be eliminated from the plastic loads. To eliminate the fifth nodal point consider equation (B-4) relating nodal point loads to nodal point displacements. In equation (B-5) substitute for

$\{u2\}_i$ from equation (B-6) and obtain

$$\{q1\}_i - [k2]_i[k4]_i^{-1}\{q2\}_i = [\tilde{k}1 - \tilde{k}2\ \tilde{k}4^{-1}\ \tilde{k}3]_i\{u1\}_i \tag{B-47}$$

which can be rewritten as

$$\{q1\}_i - [k2]_i[k4]_i^{-1}\{q2\}_i = [\tilde{I} \mid \tilde{k}2\ \tilde{k}4^{-1}]_i \begin{Bmatrix} \tilde{q}1 \\ \hline \tilde{q}2 \end{Bmatrix}_i \tag{B-48}$$

or

$$\{q''\}_i = [\tilde{I} \mid \tilde{A}] \begin{Bmatrix} \tilde{q}1 \\ \hline \tilde{q}2 \end{Bmatrix}_i \tag{B-49}$$

where

$$\{q''\}_i = \{q1\}_i - [k2]_i[k4]_i\{q2\}_i \tag{B-50}$$

$$\tilde{A} = \tilde{k}2\ \tilde{k}4^{-1} \tag{B-51}$$

and

$\tilde{I}$ - an (8 x 8) identity matrix

Making use of equation (B-49) to eliminate two loads from $\{b\}_i$ gives

$$\{b''\}_i = [\tilde{I} \mid \tilde{A}]_i\{b\}_i \tag{B-52}$$

where

$\{b''\}_i$ - (8 x 1) reduced element load vector

$\{b\}_i$ - (10 x 1) complete element load vector

Substituting for $\{b\}_i$ from equation (B-46) gives

$$\{b''\}_i = [I \mid A]_i \begin{bmatrix} \tilde{F}H1 \\ \hline \tilde{F}H2 \end{bmatrix}_i \{\varepsilon^{pt}\}_i \tag{B-53}$$

or

$$\{b''\}_i = [\tilde{FH1} + \tilde{A}\ \tilde{FH2}]_i \{\epsilon^{pt}\}_i \tag{B-54}$$

where

$\tilde{FH1}$ is the first eight rows of $[FH]_i$

and

$\tilde{FH2}$ is the last two rows of $[FH]_i$

Evaluating $\tilde{A}\ \tilde{FH2}$ gives an (8 x 4) matrix which can be added to $[FH1]_i$.

Redefining $[FH]_i$

$$[FH]_i = [FH1]_i + [\tilde{A}\ \tilde{FH2}]_i \tag{B-55}$$

and

$$\{b''\}_i = [FH]_i \{\epsilon^{pt}\}_i \tag{B-56}$$

$[FH]_i$ is stored on magnetic disk when $\{b\}_i$ is first calculated. When increments of plastic strain are calculated the proper loading for a quadrilateral can be calculated by multiplication of the plastic strain vector with $[FH]_i$. The matrix $[FH]_i$ for a triangle will convert plastic strain directly to nodal point loads but will not eliminate any nodal equations.

APPENDIX C

COMPUTER DATA INPUT

A. Identification

ANALYSIS OF PLANE AXISYMMETRIC SOLIDS

Programmed - V. Allen and H. D. Keith

University of Missouri - Rolla

B. Purpose

The purpose of this computer program is to determine deformations and stresses within plane or axisymmetric structures. The effects of displacement or stress boundary condition, concentrated loads, gravity forces and temperature changes are included. Nonlinear material properties are included by incremental loading and a successive approximation technique, which utilizes the initial strain approach.

C. Computer Program Input

The first step in the structural analysis of a solid is to select a finite element representation of the two-dimensional cross-section of the body. Elements and nodal points are then numbered in two numerical sequences each starting with one. The following group of punched cards numerically define the two-dimensional structure to be analyzed.

1. Identification Card - (72H). Columns 1 to 72 of this card contain identification information or title to be printed with results.

2. <u>Control Card - (4I5, 4F10.2, I5, F10.2, I5)</u>.

Columns 1 - 5 Total number of nodal points (450 maximum)

6 - 10 Total number of elements (450 maximum)

11 - 15 Total number of different materials (12 maximum)

16 - 20 Number of boundary pressure cards (450 maximum)

21 - 30 Axial acceleration of the entire body in the Z-direction

31 - 40 Angular velocity of the entire body

41 - 50 Reference temperature (stress free temperature)

51 - 60 Load increment size as a fraction of yield load

65 Geometry Option 1 . . . Plane

0 . . . Axisymmetric

66 - 75 Convergence criteria in percent of total element strain

80 Data test option 1 . . . Test data

0 . . . Run complete program

3. <u>Material Property Information</u>

The following group of cards must be supplied for each different material:

First Card - (2I5, 2F10.0)

Columns 1 - 5 Materials identification - any number from 1 to 12

6 - 10 Number of different temperatures for which properties are given (4 maximum)

Following Cards - Three cards for each temperature listed in columns 6 - 10 of the first card of this group:

Second Card - (5F10.0)

Columns 1 - 10 Temperature

11 - 20 Modulus of elasticity - E_z

21 - 30 Poisson's ratio - μ

31 - 40 Coefficient thermal expansion - α

41 - 50 Yield stress - σ_y

The third card lists values of the stress and the fourth card lists strain as shown in Figure 18.

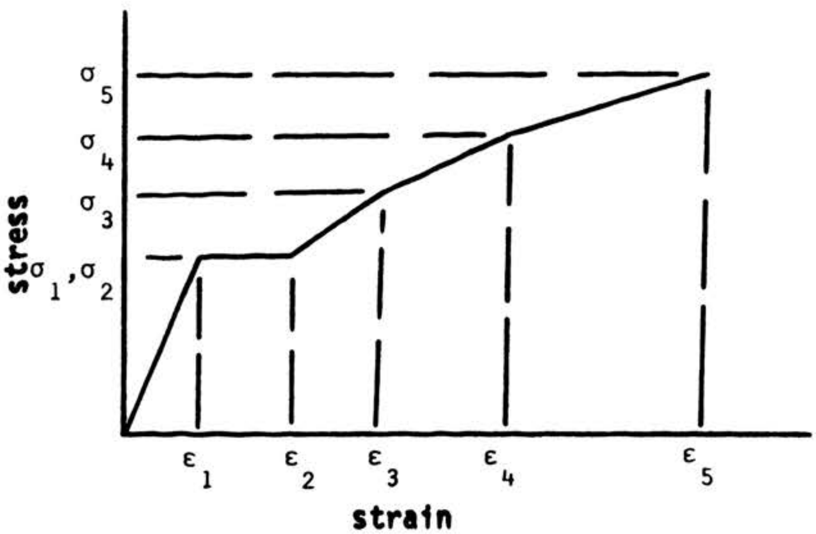

Figure 18. Uniaxial Stress Strain Curve

Card 3 Stress - (4F10.0)

Columns 1 - 10 Stress σ_2

11 - 20 Stress σ_3

21 - 30 Stress σ_4

31 - 40 Stress σ_5

Card 4 Strain - (4F10.0)

Columns 1 - 10 Strain ε_2

11 - 20 Strain ε_3

21 - 30 Strain ε_4

31 - 40 Strain ε_5

Note: One value of stress and strain must be specified for which the strain is larger than the maximum strain occurring in the solution of the problem.

4. Nodal Point Cards - (I5, F5.0, 5F10.0). One card for each nodal point with the following information:

Columns 1 - 5 Nodal point number

6 - 10 Number which indicates if displacement or forces are to be specified

11 - 20 R - ordinate

21 - 30 Z - ordinate

31 - 40 XR

41 - 50 XZ

51 - 60 Temperature

If the number in columns 6 - 10 is

0.0 XR is the specified R-load and
XZ is the specified Z-load

1.0 XR is the specified R-displacement and
XZ is the specified Z-load

2.0 XR is the specified R-load and
XZ is the specified Z-displacement

3.0 XR is the specified R-displacement and

XZ is the specified Z-displacement

All loads are considered to be total forces acting on a one radian segment.

Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals along a straight line between the defined nodal points. The necessary temperatures are determined by linear interpolation. The boundary code (columns 6 - 10), XR, and XZ are set equal to <u>zero</u> for all generated nodal points.

5. <u>Element Cards - (6I5)</u>. One card for each element.

Columns 1 - 5	Element number	1. Order nodal points counterclockwise around element.
6 - 10	Nodal Point I	2. Maximum difference between nodal point I.D. must be less than 30.
11 - 15	Nodal Point J	
16 - 30	Nodal Point K	
21 - 25	Nodal Point L	
26 - 30	Material Identification	

Element cards must be in element number sequence. If element cards are omitted, the program automatically generates the omitted information by incrementing by one the preceding I, J, K, and L. The material identification code for the generated cards is set equal to the value given on the last card. <u>The last element card must always be supplied.</u>

Triangular elements are also permissible, and are identified by repeating the last nodal point number (i.e., I, J, K, K).

6. Pressure Cards - (2I5, 1F10.0). One card for each boundary element which is subjected to a normal pressure.

Columns 1 - 5 Nodal Point I

6 - 10 Nodal Point J

11 - 20 Normal Pressure

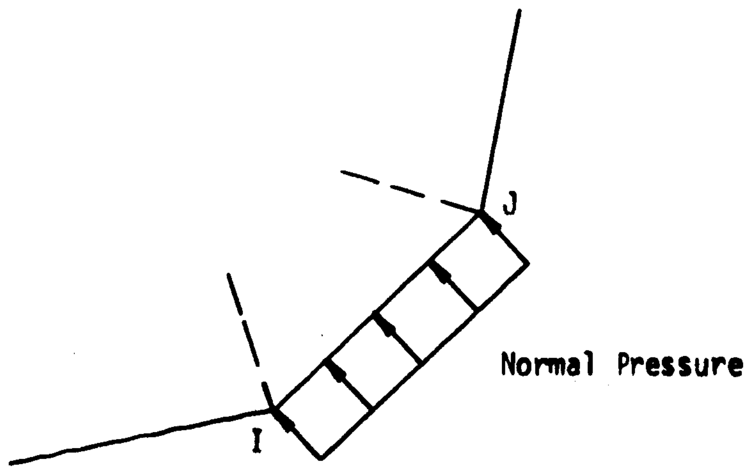

Figure 19. Pressure Boundaries

As shown above, the boundary element must be on the left as one progresses from I to J. Surface tensile force is input as a negative pressure.

D. Additional Remarks on Use of the Program

The previous section contains a schematic description of the program input. The purpose of this section is to explain in greater detail the various options of the program.

1. Output Information. The following information is developed and printed by the program:

   a. Reprint of input data

   b. Nodal point displacements

   c. Stresses at the center of each element

   d. Plastic elements and convergence index

   e. Element mean stress

   f. Element principal stresses and principal direction

   g. Element equivalent stress

   h. Largest effective stress

   i. Elastic solution for total load

   j. Initial yield solution and solutions at increments above the initial yield load until the total load solution has been obtained

2. Material Properties. Material properties vs. temperature are input for each material in tabular form. The properties for each element in the system are then evaluated by interpolation. The mass density of the material is required only if acceleration loads are specified. Listing of the coefficient of thermal expansion is necessary only for thermal stress analysis.

3. Skew Boundaries. If the number in columns 6 - 10 of the nodal point cards is other than 0.0, 1.0, 2.0 or 3.0 it is interpreted as the magnitude of an angle in degrees. This angle is shown below.

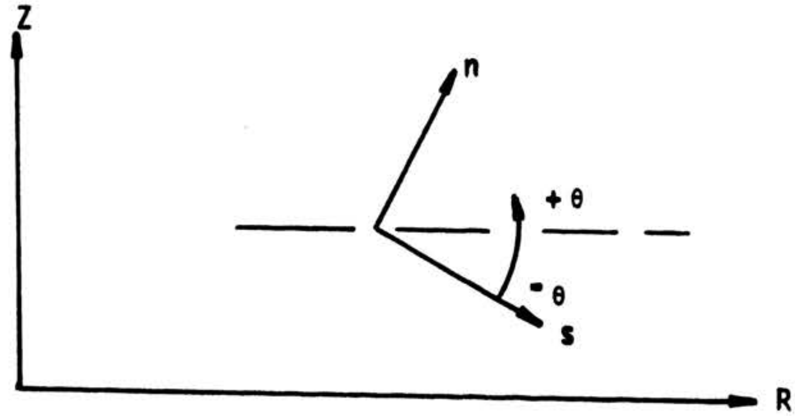

Figure 20. Skew Boundaries

The terms in columns 31 - 50 of the nodal point card are then interpreted as follows:

XR is the specified load in the s-direction

XZ is the specified displacement in the n-direction

The angle θ must always be input as a negative angle and may range from -.001 to -180.0 degrees. Hence, +10.0 degrees is the same as -170.0 degrees. The displacements of these nodal points which are printed by the program are

u_r - the displacement in the s-direction

u_z - the displacement in the n-direction